NEW SECOND COURSE IN ALGEBRA

HAWKES - LUBY - TOUTON

ENLARGED EDITION

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MATHEMATICAL TEXTS

FOR SCHOOLS

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NEW SECOND COURSE IN ALGEBRA

ENLARGED EDITION

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PREFACE

The New Second Course in Algebra is a revision of the Revised Second Course in Algebra, and is intended for substantially the same class of students. It is published in two editions, the regular edition and the present Enlarged Edition.

The first part of the Enlarged Edition is identical in most respects with the regular edition. The later chapters, however, include subjects which are either further developments of topics treated less fully in the regular edition, or are so closely related to intermediate algebra as to make their presentation to the student desirable at this time. Several hundred supplementary exercises and problems, paralleling the chapters of the book, provide sufficient additional drill to meet the needs of any teacher.

As in the regular edition, every effort has been made to have the problems stimulating and also representative of actual situations which are of frequent occurrence in science and engineering. Rather more emphasis has been placed upon the explicit application of the formula than appeared in the earlier editions of this book. In a certain sense a large portion of the work in any algebra might properly be included under the caption of the formula, but recent usage has narrowed the application of the term.

It will be found that the selection and treatment of material conform to the recommendations of the various examining bodies whose reports and requirements seem to be of importance.

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NEW SECOND COURSE IN ALGEBRA

CHAPTER I

REVIEW OF FUNDAMENTAL OPERATIONS

1. Algebraic expressions. A letter or a group of number symbols such as $3x - 4$, ab^2, $\frac{x - 2m}{a + b}$, or $4a(x - y)$ is called an *algebraic expression.*

While 2×5 always equals 10, $l \times w$ may equal 5, 6, 10, 20, etc., depending on the values assigned to l and w respectively. The fact that an arithmetic expression always represents the same number while the value of an algebraic expression may change when different values are assigned to the letters involved constitutes one of the chief differences between arithmetic and algebra.

2. Order of fundamental operations. The numeric value of an arithmetic or an algebraic expression involving signs of addition, subtraction, multiplication, and division depends on the order in which the indicated operations are performed. It is understood that

In a series of operations involving addition, subtraction, multiplication, and division, first the multiplications and divisions shall be performed in the order in which they occur. Then the additions and subtractions shall be performed in the order in which they occur or in any other order.

Within any parenthesis the preceding rule applies.

If the multiplication of two or more numbers or number symbols is indicated, as in $5a$ or $3x^2$, without any symbol of multiplication, it is customary to think of the indicated multiplication as already having been performed. Thus $3x \div 2a = \frac{3x}{2a}$ not $\frac{3x}{2} \cdot a$.

EXERCISES

Simplify:

1. $8 - 3 + 10 - 4$.
2. $7 + 15 \div 3 - 8$.
3. $5 \cdot 6 \div 2 + 7$.
4. $30 \div 3 \div 5$.
5. $40 \div 4 \div 2 + 2 - 3 + 4 \cdot 2$.
6. $(21 - 3 \cdot 4)(17 - 5 \cdot 2) - 3 \cdot 7$.
7. $18 - 8 \cdot 2 \div 4 + 7 \cdot 3 - 6 \div 2$.

Find the numeric value of:

8. $3x + 4 \div 2 - x$ if $x = 4$.
9. $(3x - 2)(x + 4)$ if $x = 2$.
10. $r^2 - 5r + 6$ if $r = 3$; if $r = 5$; if $r = 1$; if $r = 12$.
11. $y^3 - 3y^2 + 4y - 20$ if $y = 0$; if $y = 1$; if $y = 4$.
12. $3p^3 + 7p^2 - 5p + 2$ if $p = 0$; if $p = 2$.
13. Does $3(5x - 2) + 14 = 4x^2 + 2x + 11$ if $x = 3$? if $x = 0$? if $x = 2$?
14. Does $\frac{(m - 1)(m + 1)}{m^2 + 2m + 1} = \frac{m - 1}{m + 1}$ if $m = 3$?

3. Terms. Algebraic expressions are often regarded as made up of parts separated by the signs of operation $+$ or $-$. Each of these parts is called a *term*.

A *binomial* is an algebraic expression of two terms.

A *trinomial* is an algebraic expression of three terms.

A *polynomial* is an algebraic expression of two or more terms.

Similar terms are terms which are alike in all respects except their coefficients.

Dissimilar terms are terms which differ in other respects than their coefficients, though the latter may be alike or different.

4. Addition. In algebra, *addition* involves the uniting of similar terms which have the same or opposite signs into one term. For this we have the following rules:

I. ***To find the sum of two or more numbers all of which have the same sign, add the numbers numerically, and prefix the common sign to the result.***

II. ***To find the sum of two numbers with unlike signs, subtract the number with the smaller numeric value from the number with the greater numeric value, and prefix the sign of the greater number to the result.***

Obviously $3 + 4 + 6 = 3 + 6 + 4 = 6 + 3 + 4$, etc. Even if we have a series of positive and negative numbers, the order in which they are combined does not affect their sum. This principle of addition is called the *Commutative Law for Addition.*

For the addition of polynomials we have the

RULE. ***Write similar terms in the same column.***

Find the algebraic sum of the terms in each column and write the results in succession with their proper signs.

EXERCISES

Add:

1. $12, -8, +4, -6, +7.$
2. $7x, 3x, -4x, 3x, -5x.$
3. $3r - 7w, 4r - 3w, 6r + 5w,$ and $-3r + 4w.$
4. $5r - 2s + 3v, -3r + 4s - 7v,$ and $4r - 6s + 2v.$

5. $5x^2 - 6xy + 3y^2$, $-7x^2 + 3xy - 4y^2$, $3x^2 + 3xy + y^2$, and $-x^2 + xy$.

6. $9a^2 - 3b + 2c$, $4a^2 + 5b - 6c + 2$, $-4a^2 + 3b + 5c + 4$, $-8a^2 - 5b - c - 6$.

7. What name is given to each 2 in $r^2 - 2r$? Distinguish between a coefficient and an exponent.

8. What is the coefficient of s in $2rs$? of b in $3ab$? of a^2 in $3a^2b$?

9. What is the coefficient of $a + b$ in $2x(a + b)$? in $5(a + b)d$? in $3a + 3b$? in $ac + bc$?

Write so that a, b, or x shall have a polynomial coefficient:

10. $ax + bx$.
11. $3a - 4ab$.
12. $4ax + 3xy - 7x$.
13. $3ab - 4b + 7by$.
14. $6bx - 3ax + 5x$.
15. $a + 3ay + a$.
16. $ax + bx + cx$.
17. $2a - a + 3ay$.
18. $2px - 3rx + x - 5x$.
19. $3ax + 2bx - 3cx$.
20. $2ab - b + bc + b$.

Write the following so that the binomial will have a binomial coefficient:

21. $(a - 2)x + (a - 2)y$.

Solution. $(a - 2)x + (a - 2)y = (a - 2)(x + y)$.

22. $(3a - 2b)3x - (3a - 2b)2y$.
23. $(3a - 2)6b - (3a - 2)7a$.
24. $4r(5a - 4c) - 3s(-4c + 5a)$.
25. $(3x - 4y)8a^2f - (-4y + 3x)7bc^4$.
26. $-13xy^3(-7ax + 6by) + 15a^4bc(6by - 7ax)$.

5. Subtraction. For the subtraction of polynomials we have the

RULE. *Write the subtrahend under the minuend, so that similar terms are in the same column.*

Then changing mentally the sign of each term of the number to be subtracted, add the similar terms algebraically.

Check. As in Arithmetic, difference + subtrahend = minuend.

EXERCISES

Subtract the first term from the second in the following:

1. $7\,ax$, $9\,ax$.
2. $33\,ay^8$, $15\,ay^8$.
3. $-6\,x^4z$, $18\,x^4z$.
4. $7\,a^4b$, $-7\,a^4b$.
5. $5\,x^2 - y^2$, $7\,x^2 - y^2$.
6. $7\,mp + 2\,rv$, $3\,rv - 6\,mp$.
7. $7\,x^4 - 3\,y^3$, $-4\,x^4 + 4\,y^3$.
8. $3\,x - 4\,y + 7\,z$, $5\,x + y - 6\,z$.

Find the expression which added to the first term will give the second:

9. $3\,m^2 - 5\,m + 3$, $6\,m^2 - 11\,m + 7$.
Check for $m = 3$.

10. $9\,x^2 - 7\,bx + b^2$, $12\,x^2 + 9\,bx - 8\,b^2$.
Check for $x = 2$, $b = 3$.

11. $7\,b^2 + 3\,bm - 3\,m^2$, $10\,b^2 - 4\,bm - 5\,m^2$.

12. From the sum of $5\,ax^2 + 7\,by^3 - z^4$, and $6\,ax^2 - 3\,by^3 - 2\,z^4$, take $7\,ax^2 + 3\,by^3 - 3\,z^4$.

13. From the sum of $7(a + b)^2$, $5(a + b)^2$, and $-3(a + b)^2$, take the sum of $-3(a + b)^2$ and $9(a + b)^2$.

14. From the sum of $3\,x^2 + 2\,xy - 5\,y^2$, $2\,x^2 - 5\,y^2$, and $x^2 - 10\,xy + 3\,y^2$, take the sum of $-5\,x^2 + 2\,xy + y^2$, $10\,x^2 + 2\,xy$, and $9\,x^2 + 5\,xy + 3\,y^2$.

6. Multiplication. In multiplying one term by another, the sign of the product, the coefficient of the product, and the exponent of each letter of the product are determined as follows:

I. *The sign of the product is* plus *if the multiplier and the multiplicand have like signs, and* minus *if they have unlike signs.*

II. *The coefficient of the product is the product of the coefficients of the factors.*

III. *The exponent of each letter in the product is determined by the general law*

$$n^a \times n^b = n^{a+b}.$$

For the multiplication of two monomials, we have the

RULE. *Obeying the rule of signs for the multiplication, write the product of the numeric coefficients followed by all the letters that occur in the factors, each letter having as its exponent the sum of the exponents of that letter in the factors.*

For the multiplication of a polynomial by a monomial, we have the

RULE. *Multiply each term of the polynomial by the monomial and write the resulting terms in succession with their proper signs.*

For the multiplication of polynomials we have the

RULE. *Multiply the multiplicand by each term of the multiplier in turn, and add the partial products.*

ORAL EXERCISES

Perform the indicated operations:

1. $(+2) \cdot (+7)$; $(+3) \cdot (-4)$; $(-5) \cdot (-7)$.

2. $(2a) \cdot (3b)$; $(5r) \cdot (2s)$; $(-2h) \cdot (+9k)$.

3. $(b^2) \cdot (b^5)$; $(x^3) \cdot (x^5)$; $(m^2) \cdot (-m^4)$.

4. $(m^5) \cdot (-m^4) \cdot (m^8)$.

5. $(r^3) \cdot (-r^5) \cdot (-r^7)$.

6. $(a^3)^2$; $(-x^2)^3$; $(-2r^3)^3$.

7. $x^n \cdot x^n$; $n^a \cdot n$; $n^a \cdot n^{a+2}$.

8. $(s^4) \cdot (s^2)^2(-s^3)^2$.

9. x^{n+1}, x^{n+2}.

10. $3ax \cdot 4a^2x \cdot 2ax^2$.

11. $2a^2b \cdot a^3b^2 \cdot 3ab^4$.

Express with one exponent:

12. $2 \cdot 2^2 \cdot 2^3$.

13. $3^2 \cdot 3^3 \cdot 3^4$.

14. $2^8 \cdot 2^4(-2)^2$.

15. $5^3 \cdot 5^4(-5)^3$.

Multiply:

16. $4(a + b)$ by $3(a + b)$.

17. $3(a + x)$ by $\frac{7}{3}(a + x)$.

18. $4(x^3 + y)$ by $\frac{2}{2}(x^3 + y)$.

19. $3(a + b + c)$ by $4(a + b + c)$.

20. $x^2(a - m + n)$ by $2x^3(a - m + n)^2$.

21. $a^2m^3n(2a^2 - 3amn - 4an^2 - m^2 + 1)$.

22. $-3r^2s^2(3r^2 + 2rs - s^2)$.

7. Arrangement. A polynomial is said to be *arranged* according to the *ascending* powers of a certain letter when the exponents of that letter in successive terms increase from left to right.

Thus, $8 + 3x - 5x^2 + 2x^4$ is arranged according to *ascending* powers of x.

Again, $y^2 - 3y + 4$ and $y^4 - 3xy^3 + 3xy^2 - x^3$ are arranged in *descending* powers of y.

Whenever it is possible, the multiplicand and the multiplier should be arranged in the same order with respect to some letter, as the necessary addition can then be more easily performed.

EXERCISES

Multiply:

1. $(3x - 5)(2x + 6)$.

2. $(3r^2 - 3r + 5)(r^2 - 5r + 6)$.

3. $(m^4 + 2m^3 - 4)(m^4 - 2m^2 - 3)$.

4. $(n^3 - n^2 + 7)(n^4 + 3n^3 - 2n^2)$.

5. $(2b^2 - 3bc + c^2)(b^2 + 5bc - 4c^2)$.
Check with $b = 2$, $c = 3$.

6. $(n^2 - nr + r^2)(n^2 - r^2 + nr)$.

HINT. Write the second factor as $n^2 + nr - r^2$, that is in descending powers of n.

7. $(n^2 - np + p^2)(n^2 + p^2 + np)$.

8. $(t^3 - t^2 + t)(at^2 + a + at)$.
Check for $t = 3$, $a = -1$.

9. $(4h^2 + 6k^2 + 9hk)(4h^2 + 6k^2 - 9hk)$.
Check for $h = 1$, $k = 2$.

10. $(at^2 - 2at^3 + 5a)(6at^4 - 2at - 4at^2)$.

11. $(m^4 + m^3 + 2m^2 - m - 1)(m - 1 + 2m^2)$.

12. $(x^n + 3x^m + 2)(x^n - x^m - 2)$.

13. $\left(2 + \frac{3m}{4} - \frac{2m^2}{3}\right)\left(1 - \frac{4m}{5} + \frac{m^2}{6}\right)$. Check for $m = 2$.

14. $(a + b + c)(a^2 + b^2 + c^2 - ab - ac - bc)$.

15. $(5x - 9x^5 + x^2 - 4x^4 - 3x^3 + 10)(-x^3 + 3x - x^2 + 2x^4)$.

16. $(x^4 - x^3 + 25 - 5x - 4x^2)(x + x^2 + 5)$.

17. $(x^{2a} + x^4 + x^2 + x^{a+2} - x^{a+1} + x^3)(x^a - x^2 + x)$.

18. Does $15(x - a) - 6(x + a) = 3(5a - 3x)$ if $x = 2a$?

19. Does $ax(a + 3) + a(10 - a^2) = x + 2$ if $x = a - 3$?

8. Division. In dividing one term by another the sign of the quotient, the coefficient of the quotient, and the exponent of each letter in the quotient are obtained as follows:

I. *The sign of the quotient is* plus *when the dividend and the divisor have like signs, and* minus *when they have unlike signs.*

II. *The coefficient of the quotient is obtained by dividing the coefficient of the dividend by that of the divisor.*

III. *The exponent of each letter in the quotient is determined by the law*

$$n^a \div n^b = n^{a-b}.$$

For the division of monomials we have the

RULE. *Divide the numeric coefficient in the dividend by the numeric coefficient in the divisor, following the rule of signs for division.*

Write after this quotient all the letters of the dividend except those having the same exponent in the dividend as in the divisor, giving to each letter an exponent equal to its exponent in the dividend minus its exponent in the divisor.

If there are any letters in the divisor unlike those in the dividend, write these under the preceding result as a denominator.

ORAL EXERCISES

Perform the indicated division:

1. $\frac{x^5}{x^3}$.
2. $\frac{x^6}{x^2}$.
3. $\frac{-a^5m^6}{a^3m}$.
4. $\frac{3a^4b^2}{ab}$.
5. $\frac{-10a^3c^5}{5ac^4}$.
6. $\frac{12m^3n^5x^6}{4m^2n^5x^2}$.
7. $\frac{a^2c - c^2}{c}$.
8. $\frac{x^{n+3}}{x^2}$.
9. $\frac{-x^{n+6} + x^{2n}}{x^n}$.

9. Division of a polynomial. The method of dividing one polynomial by another is expressed in the

RULE. *Arrange both the dividend and the divisor according to the descending (or ascending) powers of some common letter, called the letter of arrangement.*

Divide the first term of the dividend by the first term of the divisor and write the result as the first term of the quotient.

Multiply the divisor by the first term of the quotient and subtract, being careful to write the terms of the remainder in the same order as those of the divisor.

To find the second term of the quotient, divide the first term of the remainder by the first term of the divisor, and proceed as before until there is no remainder, or until the remainder is of lower degree in the letter of arrangement than the divisor.

Check. Quotient $\times$ *Divisor* $=$ *Dividend.*

The following numeric check for division is often useful.

RULE. *Assign a small numeric value to the letter or letters of the dividend, divisor, and quotient. Then divide the numeric value of the dividend by that of the divisor. The result should equal the numeric value of the quotient.*

EXERCISES

Perform the indicated division:

1. $(2\,m^2 - 5\,m + 3) \div (2\,m - 3)$.

Check for $m = 3$.

2. $(x^2 - 7\,x + 12) \div (x - 3)$.

Check with $x = -2$.

3. $(6\,m^2 + 10\,mr - 4\,r^2) \div (m + 2\,r)$.

4. $(18\,r^2 + 69\,rt - 165\,t^2) \div (9\,r - 15\,t)$.

5. $(8\,m^3 - 10\,m^2 - 13\,m + 15) \div (2\,m - 3)$.

6. $(6\,a^3 + 6\,a^2 - 28 - 26\,a) \div (2\,a + 4)$.

7. $(6x^6 - 5x^4 + 25x^3 - 17x^5) \div (5x^2 - 2x^3)$.

8. $(s^6 - 7s^3 - 8) \div (s^2 + 4 + 2s)$.

Check with $s = 2$.

9. $(x^3 - 5a^2x + 2a^3) \div (x^2 + 2ax - a^2)$.

10. $(2x^4 - 12x^2 - 2 + 11x - 7x^3) \div (1 - 3x - 2x^2)$.

11. $(23s^2 - 13s^3 + 2s^4 - 60 - s) \div (5 + 3s - s^2)$.

12. $(m^3n^3 + 8p^3 + 125 - 30mnp) \div (mn + 2p + 5)$.

13. $(x^3 + y^3 + z^3 - 3xyz) \div (x + y + z)$.

14. $(32x^4 - 60 - 2x - 104x^3 + 92x^2) \div (5 + 6x - 4x^2)$.

15. $(a^2 - 2ab + b^2 - 9x^2) \div (3x + b - a)$.

16. $(m^4n^4 - 2m^2n^2p^2 + p^4 - 16q^4) \div (m^2n^2 - p^2 + 4q^2)$.

17. $8b^2c^2 + 1 - 16c^4 - b^4 \div 4c^2 - b^2 - 1$.

18. $m^5 + m^3 + 8m^2 + 8 \div m^2 - 2m + 4$.

19. $3r^4 - 23r^3 - 7r^2 + 42r - 10 \div 3r^2 + r - 5$.

20. $2x^5y^{10} - 9x^4y^8 + 15x^3y^6 - 27x^2y^4 + 20xy^2 - 4 \div 2x^2y^4 - 7xy^2 + 2$.

21. $m^6 - 7m^4r + 3m^2r^2 - 21r^3 \div m^4 + 3r^2$.

22. $(x^{4a+4} - 16x^4) \div (x^{a+1} + 2x)$.

23. $(a^{4n} + a^4 + a^{2n+2}) \div (a^2 + a^{2n} - a^{n+1})$.

10. Parentheses. The removal of a parenthesis preceded by a plus sign is governed by the

PRINCIPLE. *A parenthesis and the plus sign before it may be removed from an expression without changing the signs of the terms which were inclosed by the parenthesis.*

The removal of a parenthesis preceded by a minus sign is governed by the

PRINCIPLE. *A parenthesis and the minus sign before it may be removed from an expression, provided the sign of each term which was inclosed by the parenthesis be changed.*

This principle is equivalent to the principle which holds in the subtraction of one polynomial from another.

When one parenthesis incloses another, either the outer or the inner parenthesis may be removed first. Usually it is best to use the

RULE. *Rewrite the expression, omitting the innermost parenthesis, changing the signs of the terms which it inclosed if the sign preceding it be minus and leaving them unchanged if it be plus.*

Combine like terms within the new innermost parenthesis.

Repeat these processes until all the parentheses are removed, and similar terms combined.

EXERCISES

Remove parentheses and simplify:

1. $2 - 1 - (3 - 1) + (2 - 3)$.
2. $5 + 6 - (-5 + 3) + (-6 - 2) - 3 + 10$.
3. $a + (a - b) - (a - 3b)$.
4. $a - (a - m) + (2m - 3a)$.
5. $3m - (p - m) - (r - p)$.
6. $8x + (3r - 8x + 2) - (2r - 3x + 2)$.
7. $x - y + 3(x - y) - 4(2x - y)$.
8. $4x - a + [-(3c - x) - (2a - 3x)]$.
9. $m - [-(m - 4) + (3r - 2m) - 6m] + 7r$.
10. $x - 4 - (s - 2x) - [3(s - x + 4) - 2(6 - 4s)]$.
11. $4t^2 - 3t - 2t(3 + 2t)$.
12. $x^2 - 4 - (x - 1)(x + 4)$.
13. $4x^2 - 3r^2 - (2r - 3x)(2x - r)$.
14. $6x - [-(b - 2c) + (3c - 4b) - (4b - 3x)]$.
15. $8b - [(3b - 4) - 8 - (3b - 4a + 1)]$.

16. $4x - 2(x - 3) - 3[x - 3(4 - 2x) + 8]$.

17. $2t^2 + 7t - (2t + 1)(3t + 2)$. Check for $t = 3$.

18. $(x - 4)(x - 3) - (x - 3)(x + 2)$.

19. $5a^2 - [6a^2 + 2a - (3a^2 - 1) + 10a] + 5(2a - 1)$.

20. $(x - b - 2) - (b + x + 2) - [(-2b - 4) + x - b]$.

21. $(x + 4)(x^2 - 4x + 16) - (x - 4)(x^2 + 4x + 16)$.

11. **Important special products.** The following forms should be memorized so that one can write or state the results without the labor of actual multiplication.

I. For the square of the sum of two terms we have the formula

$$(a + b)^2 = a^2 + 2ab + b^2.$$

This expressed verbally is:

The square of the sum of two terms is the square of the first term plus twice the product of the two terms plus the square of the second term.

II. For the square of the difference of two terms we have the formula

$$(a - b)^2 = a^2 - 2ab + b^2.$$

This expressed verbally is:

The square of the difference of two terms is the square of the first term, minus twice the product of the two terms, plus the square of the second term.

III. For the product of the sum and the difference of two terms we have the formula

$$(a + b)(a - b) = a^2 - b^2.$$

This expressed verbally is:

The product of the sum and the difference of two terms equals the difference of their squares taken in the same order as the difference of the terms.

IV. For the product of two binomials having a common term we have the formula

$$(x + a)(x + b) = x^2 + (a + b)x + ab.$$

This expressed verbally is:

The product of two binomials having a common term equals the square of the common term, plus the algebraic sum of the unlike terms multiplied by the common term, plus the algebraic product of the unlike terms.

V. The square of the polynomial $a + b - c$ gives the formula

$$(a + b - c)^2 = a^2 + b^2 + c^2 + 2\,ab - 2\,ac - 2\,bc.$$

This expressed verbally is:

The square of any polynomial is equal to the sum of the squares of each of the terms plus twice the algebraic product of each term by every term that follows it in the polynomial.

VI. The cube of the binomial $a + b$ gives the formula

$$(a + b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3.$$

This expressed verbally is:

The cube of the sum of two numbers equals the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

VII. Similarly,

$$(a - b)^3 = a^3 - 3\,a^2b + 3\,ab^2 - b^3.$$

This can be expressed verbally in a manner similar to VI.

ORAL EXERCISES

State the results of the indicated multiplication:

1. $(x + 3)^2$.
2. $(a + 8)^2$.
3. $(3h + 1)^2$.
4. $(2r + 5)^2$.
5. $(3m + 2)^2$.
6. $(4k + 3)^2$.
7. $(a - 3)^2$.
8. $(2n - 1)^2$.
9. $(3x - 5)^2$.
10. $(8s - .3)^2$.
11. $(x + 7a)^2$.
12. $(2x - 3y)^2$.
13. $(8h - 5k)^2$.
14. $(7x - 6t)^2$.
15. $(x^2 - x)^2$.
16. $(a^2 + 3a)^2$.
17. $(m + 6n)^2$.
18. $(3r - 4s)^2$.
19. $(mn - 3qr)^2$.
20. $(.5x + y)^2$.
21. $(.6x + .2y)^2$.
22. $(5x + .03y)^2$.
23. $(a^z + 1)^2$.

Solution. $(a^z + 1)^2 = a^{2z} + 2a^z + 1$.

24. $(x^z + y^b)^2$.
25. $(x^{2b+1} + 3)^2$.
26. $(x^{3a+1} + y^b)^2$.
27. $(\frac{1}{2}x + 4)^2$.
28. $(\frac{1}{2}x + 3)^2$.
29. $(\frac{1}{3}r - \frac{1}{2}s)^2$.
30. $(x - a)(x + a)$.
31. $(x - 5)(x + 5)$.
32. $(2x - 1)(2x + 1)$.
33. $(ax + 3)(ax - 3)$.
34. $(4r - 3s)(4r + 3s)$.
35. $(ar - b)(ar + b)$.
36. $(6a + b)(b - 6a)$.
37. $(3m + 2n)(3m - 2n)$.
38. $(x + 3)(x + 4)$.
39. $(x + 2)(x + 5)$.
40. $(x + 1)(x + 7)$.
41. $(r - 3)(r - 5)$.
42. $(r - 2)(r - 7)$.
43. $(m - 3)(m + 4)$.
44. $(m + 6)(m - 7)$.
45. $(x + 2)(x - 9)$.
46. $(x - 5)(x + 10)$.
47. $(ax - 3)(ax + 5)$.
48. $(2x - 1)(2x + 3)$.
49. $(3y + 1)(3y + 4)$.

50. $(4n - 2)(4n + 3)$.
51. $(a^2 - 3a)(a^2 + 4a)$.
52. $(a + b + c)^2$.
53. $(b + c + x)^2$.
54. $(a + b - x)^2$.
55. $(a - b + x)^2$.
56. $(r + s + 1)^2$.
57. $(r - s + 1)^2$.
58. $(a + c + 2)^2$.
59. $(a + c + 6)^2$.
60. $(a + c)^3$.
61. $(a + 2)^3$.
62. $(a + 3)^3$.
63. $(a - x)^3$.
64. $(x - 2)^3$.
65. $(x + 1)^3$.
66. $(1 - a)^3$.
67. $(h - 3)^3$.
68. $(h + 2r)^3$.

EXERCISES

Find the following products :

1. $[(x + y) + 1][(x + y) - 1]$.
2. $[(x + a) + 3][(x + a) - 3]$.
3. $[(x - r) + 3][(x - r) - 3]$.
4. $[(m + 3) + x][(m + 3) - x]$.
5. $[(2a - b) + c][(2a - b) - c]$.
6. $(x - 3r + 8)(x + 3r - 8)$.
7. $[(h + s) + (a + b)][(h + s) - (a + b)]$.
8. $[(x + y + z) + (m + r)][(x + y + z) - (m + r)]$.
9. $(.5x + 7y)(.5x - 7y)$.
10. $(1.2h + 1.3s)(1.2h - 1.3s)$.

PROBLEMS

1. A square field has a side $2s$ feet. What is the decrease in the area if its side is decreased 6 feet?

2. From each corner of a square piece of tin of side c inches a square of side x inches is cut. By turning up the sides an open box is formed. Show that $c^2 - 4x^2$ inches is the area of the box.

3. Express the area $c^2 - 4x^2$ inches of Problem 2 as the product of two binomials.

4. Using the results of Problem 3 find by a short method the area of the inside of the box if $c = 15$ and $x = 2$; if $c = 80$ and $x = 5$; if $c = 70$, $x = 15$.

5. The dimensions of a rectangular box are n, $n + 2$, $n + 5$. Express (a) the sum of the edges, (b) the total outer surface, (c) the volume of the box.

6. An entrance hall with a center fountain is in the shape of a hollow square, as in the accompanying figure. What is the area of the floor surface?

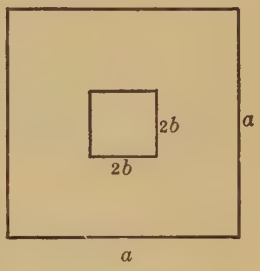


FIG. 1

7. Find the area of the hall in Problem 6 if $a = 9$ yards and $b = 2$ feet.

8. Show that the area of Figure 2 illustrates the product $a^2 - b^2$.

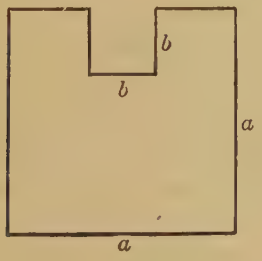


FIG. 2

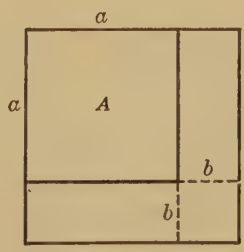


FIG. 3

9. What product is illustrated by the area of A in Figure 3?

10. Find the total area of Figure 3.

11. Find the area of Figure 3 if $a = 10, b = 6$; if $a = 45$, $b = 8$.

12. If two equal boxes of dimensions b, $b - 3$, and $b + 7$ are placed end to end, find (*a*) the sum of the outer edges, (*b*) the outer surface, and (*c*) the combined volume.

13. The formula for changing a Centigrade thermometer reading to Fahrenheit is $F = \frac{9}{5} C + 32$.

What is the Fahrenheit reading when the Centigrade reading is (*a*) 55°? (*b*) 18°? (*c*) 75°? (*d*) 100°? (*e*) 32°? (*f*) $-15°$?

14. The area of a circle is given by the formula πr^2, in which $\pi = 3.1416$ and r equals the radius of the circle. What is the area of a circular flower bed of radius 5 feet? of radius 7 feet? of radius r feet?

15. What is the area of a circular ring left after cutting a circle of radius r from a circle of radius R?

16. What is the area of a cross-section of pipe whose diameter is 12 inches, if the inside diameter is 8 inches?

17. What is the area of a cross-section of a concrete standpipe whose outside diameter is 30 feet, if the masonry is 4 feet in thickness?

18. The volume of a cylinder is found by using the formula $\pi r^2 h$, in which r equals the radius of the circular base and h is the height of the cylinder. What is the volume of a cylinder if r equals 10 and $h = 15$?

19. What is the volume of water contained in the standpipe of Problem 17 if $h = 200$ feet?

How many gallons would the standpipe hold, allowing $7\frac{1}{2}$ gallons to the cubic foot?

20. The surface of a cylinder is $2\pi r(h + r)$, in which r is the radius of the base and h is the height of the cylinder. Find the amount of tin in 10 cases of cylindrical cans which have a circular base of radius 2 inches and a height of 4 inches. (A case contains 144 cans.)

21. Find the total surface of a cylinder of radius $a + 7$, height $a - 4$. Use $\pi = \frac{22}{7}$.

22. Find the volume of a cylinder of radius $r - 7$ and height $r + 7$.

23. The formula for the surface of a sphere is $4\pi r^2$. What is the surface of a sphere of radius 5 inches? 7 inches? 10 feet?

24. The dome of St. Peter's, Rome, is in the form of a hollow hemisphere 140 feet in diameter (inside measurements). How many square yards of gold foil would be needed to cover the inside of the dome?

25. Compare the surface of a sphere of radius $3r$ with the surface of a sphere of radius r.

26. Compare the surface of a sphere of 8 inches radius with the sum of the surfaces of 128 balls of radius $\frac{1}{2}$ inch.

27. What is the volume of a sphere of 1 foot radius? of 5 foot radius? ($V = \frac{4}{3}\pi r^3$.)

28. How many spherical balls of $\frac{1}{2}$ inch radius can be made from a spherical ball of lead of radius 12 inches?

29. What is the volume of a sphere whose radius is $2r$ inches? $r - 7$ inches?

30. Compare the areas of two circles of radii m and $2m$ respectively.

31. Compare the volumes of two spheres of radii m and $2m$ respectively.

CHAPTER II

LINEAR EQUATIONS IN ONE UNKNOWN

12. Definition of an equation. An equation is a statement of the equality between two equal numbers or number symbols.

Thus $r(r - 5) = r^2 - 5r$ and $x - 3 = 5$ are equations.

Equations are of two kinds — *identities* and *equations of condition*.

13. Identities. An arithmetic or an algebraic identity is an equation in which either the two members are alike, term for term, or become so if the indicated operations are performed.

Thus $12 + 2 = 2 \cdot 7$; $(m + n)(m - n) = m^2 - n^2$ are identities, for in each, if the indicated operations be performed, the two members become precisely alike.

An identity involving letters is true for *any* set of numerical values of the letters in it.

Thus the identity $x(r + 2) = xr + 2x$ becomes $3(7 + 2) = 21 + 6$ or $27 = 27$, when, for example, $x = 3$, and $r = 7$.

14. Equation of condition. An equation which is true only for certain values of a letter in it, or for certain sets of related values of two or more of its letters, is an *equation of condition*, or simply an equation.

Thus $3x - 5 = 7$ is true for $x = 4$ only; and $2x - y = 10$ is true for $x = 6$ and $y = 2$ and for many other pairs of values

for x and y. There are, however, many pairs of values of x and y, such as $x = 3$ and $y = 4$, for which this equation is not true.

15. Satisfying an equation. A number or a literal expression which, being substituted for the unknown letter in an equation, changes it to an identity, is said to **satisfy** the equation.

Thus $z = 3\,b$ satisfies the equation $2\,z - b = 5\,b$, for on substituting $3\,b$ for z we have $2 \cdot 3\,b - b = 5\,b$, or $5\,b = 5\,b$, which is an identity.

After the substitution is made it is usually necessary, as above, to simplify each member before the identity becomes apparent.

16. Root of an equation. A **root of an equation** is any number or number symbol which satisfies the equation.

Thus 5 is the root of the equation $2\,r + 7 = 17$, for, when substituted for r, it satisfies the equation.

17. Axioms. An **axiom** is a statement the truth of which is accepted without proof. Some of the axioms most frequently used in algebra are

AXIOM I. *If the same number is added to each member of an equation, the result is an equation.*

AXIOM II. *If the same number is subtracted from each member of an equation, the result is an equation.*

AXIOM III. *If each member of an equation is multiplied by the same number, the result is an equation.*

AXIOM IV. *If each member of an equation is divided by the same number (not zero), the result is an equation.*

Each of these axioms is used in the solution of the

EXAMPLE

Solve $3x - \frac{2}{3} = \frac{10}{3} + 2x$.

Solution. $3x - \frac{2}{3} = \frac{10}{3} + 2x$ (1)

Multiplying (1) by 3, (Ax. III)

$9x - 2 = 10 + 6x$ (2)

Adding 2 to each member of (2), (Ax. I)

$9x = 12 + 6x$ (3)

Subtracting $6x$ from each member of (3), (Ax. II)

$3x = 12$ (4)

Dividing (4) by 3, (Ax. IV)

$x = 4$ (5)

Check. Substituting 4 for x in (1), we have

$12 - \frac{2}{3} = \frac{10}{3} + 8$ or $\frac{34}{3} = \frac{34}{3}$

Since substituting 4 for x satisfies (1), 4 is the root of (1).

18. Transposition. By the application of Axiom I to (2), in the example of the preceding section, the term -2 is removed from the first member and $+2$ is added to the second member. Again, by applying Axiom II to (3), the term $+6x$ is removed from the second member and $-6x$ is combined with the first member.

It thus appears that a term may be removed from one member of an equation, provided the same term with its sign changed from + to − or from − to + is written in or combined with the other member. This process is called **transposition.**

Hereafter, in order to simplify an equation, instead of subtracting a number from each member or adding a

number to each member, as illustrated in the foregoing example, the student should use transposition, since it is usually more rapid and convenient. He should clearly understand, however, that the *transposition of a term is really the subtraction of that term from each member of the equation.*

19. Equivalent equations. Two or more equations in one unknown, even if of very different form, are **equivalent** if all are satisfied by every value of the unknown which satisfies any one of them.

Equations (2), (3), (4), and (5) of section 17 are each equivalent to equation (1) and to each other, for all are satisfied by the same value of the unknown, $x = 4$.

Of the four axioms or assumptions of section 17 we shall make constant use. If the " same number " referred to in each is expressed arithmetically, the result is always an equation *equivalent* to the *original* one. Further, if *identical expressions involving the unknown* be added to or subtracted from each member of an equation, the resulting equation is equivalent to the first. If, however, both members of an equation be multiplied by identical expressions containing the unknown, the resulting equation *may not* be equivalent to the original one.

Multiplying each member of the equation $x - 5 = 2$ by $x - 2$, we get $x^2 - 7x + 10 = 2x - 4$, or $x^2 - 9x + 14 = 0$. Now this last equation has the roots 2 and 7, whereas the given equation has the root 7 only. Here the root 2 was introduced by multiplying the given equation by $x - 2$. Results obtained from the use of Axiom III with multipliers which contain an unknown should always be carefully checked. When a root is obtained which does not satisfy the original equation, this root should be rejected.

When the divisor contains the unknown, the use of Axiom IV may result in the loss of a root which the process of checking will not discover.

For example, if each member of $x^2 - 25 = x + 5$ is divided by $x + 5$, the result is $x - 5 = 1$, whence $x = 6$. But $x = -5$ also satisfies $x^2 - 25 = x + 5$. The root -5 was lost by dividing by $x + 5$.

If *both members of an equation are divided by a factor containing an unknown,* this factor should be set equal to zero. The root thus obtained is a root of the given equation.

With these and with certain other rare exceptions which will be noted later, the application of the axioms will produce an equation equivalent to the given one.

For solving equations in one unknown which do not involve fractions we have the

RULE. *Free the equation of any parentheses it may contain. Transpose and solve for the unknown involved.*

Reject all values for the unknown which do not satisfy the original equation.

Checking the solution of an equation is often called testing or **verifying** the result. For this we have the

RULE. *Substitute the value of the unknown obtained from the solution in place of the letter which represents the unknown in the original equation. Then simplify each member of the resulting identity until the two members are seen to be identical.*

If the correct substitution of the root for the unknown does not satisfy the equation, an error has been made in the solution.

EXERCISES

Solve and check:

1. $5x + 1 = 2x + 7$.
2. $1 + r + 5r + 17 = 0$.
3. $6x - 10 = 2x + 14$.
4. $2k - 3 = 5k + 12$.
5. $3h + 2h - 5 = 15 - 2h + 4 - h$.
6. $15x = 3(2x - 5)$.
7. $3(2x - 1) - (5x - 1) = 0$.
8. $3(h + 2) - 5(2h - 3) = 0$.
9. $6(4t - 5) - 11(2t - 3) = 0$.
10. $4(4x - 1) + 3 - 2(3 + x) = 0$.
11. $4(x - 2) + 3(2 - x) - 3x = 6(x + 1)$.
12. $(x + 1)(x - 2) = x^2 + 3$.
13. $(4x - 5)(2x - 6) - (4x - 6)(2x - 1) = 0$.
14. $x^2 - (x - 1)(x + 2) = 4x + 7$.
15. $2r(r - 3) = (2r + 5)(r - 3)$.
16. $(x + 3)^2 + 40 = (x + 5)^2$.
17. $(x + 4)^2 - (2 - x)^2 = 84$.
18. $(1 - x)(x + 2) + (x + 3)(x + 4) = 2$.
19. $(r - 3)(r + 2) + (2r - 1)(r - 5) = (3r - 6)(r + 2) - 1$.
20. $(x + 4)(x + 3) = (x + 2)(x + 1) + 42$.
21. $5(x - 3) + 3(8 - x) + 29 = 7(6 - x) + 50$.
22. $3(x+2)(x-4) + 5(x-1)(x+3) = 4(2x-1)(x-2) + 1$.
23. $3(k-7) + 5(k+4)(k+2) = k^2 + 4(k+3)(k+1) + 41$.
24. $(2x-2)(x+4) + (x-6)(2x-3) = (4x+1)(x+1) - 5$.
25. $(k - 3)^2 + 3(k - 4)^2 = 4(k - 5)^2 - 3$.

Solve for x or y:

26. $x - 2a = 4a - x$.

27. $c - x = x - c$.

28. $8s - x = x - 4r$.

29. $ax - 2ab = 4ab - ax$.

30. $3ax - 5ac = 3ac - ax$.

31. $2klh - kh^2 = kl^2 - klx + khx$.

32. $ax - a^2 + 5a = 6 + 3x$.

33. $acy + 2c - a^2c = 2cy - 2c$.

34. $x(c^2 + 25)(c - 5) = c^4 - 625$.

35. $yh^2 - yhk = h^3 - hk^2$.

36. $aby + ay + a = ab^3 - ab^2y$.

37. $6c^2 - 12c + 4x = c^3 - c^2x + 4cx - 8$.

38. $a^3 - c^3 = ax - cx$.

39. $ya - a^5 = yb - b^5$.

20. Solution of problems. In the solution of problems involving the use of simple equations the following steps are necessary.

1. *Read the problem carefully and find the statement which will later be expressed in an equation.*

2. *Represent the unknown numbers by means of numerals and letters.*

3. *Express the conditions stated in the problem as an equation involving these symbols.*

4. *Solve the equation.*

5. *Check, by substituting in the problem the values found for the unknowns.*

In the preceding sentence the words " in the problem " are of importance, for substituting the values found back into the equation would not detect any errors made in translating the words of the problem into the equation.

PROBLEMS

1. The sum of two consecutive numbers is 1175. Find the numbers.

2. The sum of four consecutive odd numbers is 1120. What are the numbers?

3. The product of two consecutive even numbers is 256 less than the square of the larger. Find the numbers.

4. The product of two consecutive odd numbers is 4688 less than the product of the next two consecutive odd numbers. Find the numbers.

5. The product of two consecutive even numbers increased by 4119 equals the product of the next two consecutive odd numbers. Find the numbers.

6. What number must be added to 8, 9, 11, and 15 in order that the product of the first and third may be 147 less than the product of the second and fourth?

7. One number exceeds another number by 7. The square of the larger number exceeds the square of the smaller number by 119. What are the numbers?

8. One pupil is four years older than another. Eight years from now the first will be $\frac{6}{5}$ as old as the second. Find their ages now.

9. Two men are 50 and 30 years of age respectively. How many years ago was the older twice the age of the younger?

10. A's age is double B's. Six years ago A's age was three times B's. How old is each?

11. One man is twice as old as another. The sum of their ages 12 years ago was $\frac{1}{2}$ the sum of their ages 12 years hence. How old are they now?

12. If each side of a square is increased by 9 feet, the area of the square will be increased by 171 square feet. What is the area of the square?

13. If one side of a square is increased by 7 feet and the other side is increased by 11 feet, the area of the resulting rectangle will be 185 square feet greater than the area of the square. Find the side of the square.

14. A certain rectangle is 10 feet longer than it is broad. If it were 3 feet shorter and 6 feet wider, its area would be 51 square feet greater. What are its length and width?

15. If each side of a square is decreased by 15 feet, the area of the square is decreased by 375 square feet. What is the side of the original square?

16. A rectangle is 17 feet narrower and 5 yards longer than a certain square. The area of the square is 675 square feet greater than the area of the rectangle. Find the dimensions of the rectangle.

17. A certain square grass plot has a strip 6 feet wide taken from all sides for a walk. The area of this border is 816 square feet. What is the side of the square?

18. Is it possible to draw a rectangle of perimeter 112 inches which has a length 5 inches greater than twice the width? If so, give its dimensions.

19. A sum of \$8.50 is in dimes, nickels, and quarters. There are three more dimes than nickels, and 9 more quarters than dimes and nickels together. How many coins of each kind are there?

20. A sum of \$23.40 consists of dollars, quarters, and dimes. If there are 6 more dimes than dollars, and the number of quarters is 4 less than twice the number of dimes, find the number of coins of each kind.

21. The sum of the digits of a certain two-digit number is 15. If the order of the digits is reversed, the number is decreased by 9. Find the number.

22. The digits of a certain three-digit number beginning at the left are consecutive odd numbers. If the sum of the digits is 21, find the number.

23. One angle of a triangle is 3 times another. The third angle is 20° more than the sum of the other two angles. If the sum of the angles of a triangle is 180°, how many degrees are there in each angle?

24. The sum of twice one acute angle of a right triangle and 4 times the other acute angle equals 330°. How many degrees are there in each acute angle?

25. The formula for the area of a circle is $A = \pi r^2$. ($\pi = \frac{22}{7}$.) If the radius of a circle is decreased by 7 inches, the area is decreased 770 square inches. Find the original radius.

26. By subtracting 2 inches from the radius of a circle whose radius is 8 inches, how much is the circumference decreased? the area?

27. Substitute r inches for 8 inches in Problem 26 and solve. Interpret your results.

28. If a circular hoop 1 foot longer than the circumference of the earth is placed about the earth so that it is everywhere equidistant from the equator and lies in its plane, how far above the equator will the hoop be?

29. Compare the result of Problem 28 with the one obtained when a similar process is carried out with a sphere 24 inches in diameter, instead of with the earth.

30. Two trains start from points 270 miles apart. What is the speed of each if one travels 2 miles an hour faster than the other and they meet in 5 hours?

31. A messenger boy on a bicycle leaves a certain place traveling at the speed of 8 miles an hour. Five hours later a man in an automobile follows and overtakes him in 3 hours. What is the speed of the automobile?

32. An automobile averaging 24 miles per hour leaves 6 hours before an express train going 45 miles an hour in the same direction as the automobile. How long after it starts will the train overtake the automobile?

33. A tree 80 feet high breaks at a point so the top of the tree strikes the ground 35 feet from the foot of the tree. What is the height of the part left standing?

34. A rectangle is 4 times as long as it is wide. A second rectangle, 7 inches shorter and 2 inches wider has a perimeter $\frac{2}{3}$ that of the first. Find the dimensions of each.

35. A marksman heard a bullet strike a target 1375 feet distant $5\frac{1}{2}$ seconds after he fired. If the velocity of sound is 1100 feet per second, what was the average velocity of the bullet?

21. Formulas. Many quantitative relationships are expressed by formulas which involve number symbols.

Thus $k = 2\pi rh$ expresses the fact that the lateral surface of a cylinder equals twice the product of π (or $\frac{22}{7}$), r, the radius of the circular base, and h, the height.

It is often necessary to transform a formula so as to express a given letter in terms of the others.

EXAMPLE

Solve $a = p(1 + rt)$ for p and r.

Solution. $a = p(1 + rt)$

Then $\frac{a}{1 + rt} = p$

Also $a = p + prt$

Then $a - p = prt$

and $\frac{a - p}{pt} = r.$

EXERCISES

1. Solve for a: $A = \frac{ab}{2}$. (Formula for area of triangle.)

2. Solve for R: $C = 2\pi R$. (Circumference of a circle.)

3. Solve for r and for t: $d = rt$.
(Relation of distance, speed, and time, in uniform motion.)

4. Solve for a and for A: $\frac{a}{A} = \frac{D}{360}$.

5. Solve for r: $I = \frac{E}{R + r}$.

6. Solve for g and t: $S = \frac{1}{2} gt^2$.

7. Solve for F: $C = \frac{5}{9}(F - 32°)$.

8. Solve for r and for n: $I = \frac{E}{R + nr}$.

9. Solve for r and for n: $I = \frac{n \cdot e}{R + nr}$.

10. Solve for a and for l: $s = \frac{n}{2}(a + l)$. (The sum of the terms in an arithmetical progression.)

11. Solve for r and for h: $V = \frac{1}{3}\pi r^2 h$. (Volume of a cone.)

12. Solve for d: $V = \frac{1}{6}\pi d^3$. (Volume of a sphere.)

13. Solve for r: $V = \frac{4}{3}\pi r^3$. (Volume of a sphere.)

14. Solve for h, b, and b': $A = \dfrac{h(b + b')}{2}$. (Area of trapezoid.)

15. Solve for r and for E: $A = \dfrac{\pi r^2 E}{180}$.

16. Solve for P_2: $\dfrac{V_1}{V_2} = \dfrac{P_2}{P_1}$.

17. Solve for a, l, and r: $s = \dfrac{rl - a}{r - 1}$.

18. Solve for t_1: $V_1 = V_0(1 - .00365\, t_1)$.

19. Solve for a, b, and f: $\dfrac{1}{a} + \dfrac{1}{b} = \dfrac{1}{f}$.

20. Solve for r: $V = \pi h^2(r - \frac{1}{3}h)$.

21. Solve for h: $V = \frac{1}{3}\pi h(r^2 + r_1^2 + rr_1)$. (Volume of the frustum of a cone.)

22. Solve for p, r, and t: $A = p(1 + rt)$. (Amount at simple interest.)

23. Solve for r, s, and r': $l = \pi s(r + r')$. (Lateral area of frustum of a cone.)

24. Find V in Exercise 11 if r equals 7 inches and h equals 15 inches.

25. Solve Exercise 16 for V_2 if $V_1 = 25$, $P_2 = 30$, $P_1 = 6$.

26. Solve Exercise 11 if $V = 1782$, $h = 9$.

27. Solve Exercise 22 for p if $A = \$6597.75$, $r = 4\frac{1}{2}\%$, $t = 3$ yrs. 6 mo.

CHAPTER III

FACTORING

22. Definition of factoring. *Factoring* is the process of finding the two or more arithmetic or algebraic expressions whose product is equal to a given expression.

In multiplication we have two factors given and are required to find their product. In division we have the product and one factor given and are required to find the other factor. In factoring, however, the problem is a little more difficult, for we have only the product given, and our experience in multiplication and division is called upon to enable us to determine the factors.

23. Rational expressions. A *rational* algebraic expression is one which can be written without the use of indicated roots of the letters involved.

Thus 3, $2a$, $5x - \sqrt{3}$, and r^2 are rational expressions. In this chapter factors which involve radicals will not be sought.

24. Integral expressions. If a rational expression can be written so as not to involve an indicated division in which an unknown letter occurs in a denominator, it is said to be integral.

Thus 2, $5k$, $\frac{a}{3}$, and $2x + 7$ are integral expressions.

25. Prime factors. An integral expression is prime when it is the product of no two rational integral expressions except itself and 1.

It must be remembered that to factor an integral expression means to resolve it into its prime factors.

The methods of this chapter enable one to factor integral rational expressions in one letter which are not prime, as well as some of the simpler expressions in two letters. No attempt is made even to define what is meant by prime factors of expressions which are not rational and integral.

There is no simple operation the performance of which makes us sure that we have found the prime factors of a given expression. Only insight and experience enable us to find prime factors with certainty.

A partial check that may be applied to all the exercises in factoring consists in multiplying together the factors that have been found. If the result is the original expression, correct factors have been found, though they may not be prime factors.

26. Polynomials with a common monomial factor. The type form is

$$ab + ac + ad.$$

Factoring, $ab + ac + ad = a(b + c + d)$.

ORAL EXERCISES

Factor:

1. $2a + 4$.	4. $a^2 + a$.	7. $8x^3 - 4x^2$.
2. $3a + 6$.	5. $2c - 6c^2$.	8. $cd + c^2d^2$.
3. $5x + 15$.	6. $9x^2 - 3x$.	9. $ax^2 - a^2x$.
10. $4cx - 8c^2$.		13. $2c + 4c^2 - 2cd$.
11. $14h - 21h^2k$.		14. $3r + 6rs + 9r^2$.
12. $5ax - 2ax^3$.		15. $4a - 10a^2 - 2a^3$.

27. Polynomials which may be factored by grouping terms and taking out a common binomial factor. The type form is

$$ax + ay + bx + by.$$

Factoring,

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) \\ &= a(x + y) + b(x + y) \\ &= (x + y)(a + b). \end{aligned}$$

EXERCISES

Separate into polynomial factors:

1. $2(a + b) + x(a + b)$.
2. $3(b + 4) + a(b + 4)$.
3. $5x(a - c) + y(a - c)$.
4. $2a(x - y) + b(x - y)$.
5. $a(c - d) - b(c - d)$.
6. $2(r - s) - b(r - s)$.
7. $h(m + 3r) - 2k(m + 3r)$.
8. $2r(5x - 4a) - 9s(5x - 4a)$.
9. $-2a(3h - k) - b(3h - k)$.
10. $r(x - s) + y(s - x)$.

HINT. Write in the form $r(x - s) - y(x - s)$, etc.

11. $2a(c - 3h) + b(3h - c)$.
12. $5r(3m - 2s) - 2t(2s - 3m)$.
13. $ac + ad + bc + bd$.
14. $ad + 2dx + 3ar + 6xr$.
15. $akr + ahr - ahs - aks$.
16. $r^2s + 2rs - 3r^2t - 6rt$.
17. $8hr + 16hs - 6kr - 12ks$.
18. $4a^2 - 4ax - ac + cx$.
19. $mr - 2r + ms - 2s + mt - 2t$.

20. $2ax - bx + cx + 2ay - by + cy$.

HINT. Group thus $(2ax - bx + cx) + (2ay - by + cy)$, etc.

21. $3r^{3x} - r^{2x} + 3r^{x} - 1$.

22. $a^{3m} + a^{2m} + a^{m} + 1$.

23. $(x - 3r)^3 + 4(x - 3r)^2$.

28. Trinomials which are perfect squares. The type form is

$$a^2 \pm 2ab + b^2.$$

Factoring, $a^2 \pm 2ab + b^2 = (a \pm b)^2$.

ORAL EXERCISES

Factor:

1. $a^2 + 2ax + x^2$.
2. $x^2 - 2xy + y^2$.
3. $m^2 - 2mr + r^2$.
4. $x^2 + 4x + 4$.
5. $r^2 - 10rs + 25s^2$.
6. $9 + 6a + a^2$.
7. $16 - 8ax + a^2x^2$.
8. $16r^2 - 24r + 9$.
9. $9x^2 - 12xy + 4y^2$.
10. $r^2 - 12rs + 36s^2$.
11. $a^{2n} - 12a^{n} + 36$.
12. $\frac{1}{4}r^4 + \frac{1}{5}r^2s^2 + \frac{1}{25}s^4$.
13. $(a + b)^2 - 2(a + b) + 1$.
14. $(r - s)^2 - 6(r - s) + 9$.
15. $4(a + 2)^2 - 12c(a + 2) + 9c^2$.
16. $9 + 6(a + x) + (a + x)^2$.
17. $16 - 8(a - 2x) + (a - 2x)^2$.
18. $x^{2a} - 4x^{a}y^{b} + 4y^{2b}$.
19. $.16r^2 - \frac{.8}{3}rs + \frac{1}{9}s^2$.
20. $.09 + .6(a + b) + (a + b)^2$.

29. A binomial the difference of two squares. The type form is

$$a^2 - b^2.$$

Factoring, $a^2 - b^2 = (a + b)(a - b)$.

More generally,
$$\begin{aligned} a^2 + 2ab + b^2 - c^2 + 2cd - d^2 &= a^2 + 2ab + b^2 - (c^2 - 2cd + d^2) \\ &= (a + b)^2 - (c - d)^2 \\ &= (a + b + c - d)(a + b - c + d). \end{aligned}$$

ORAL EXERCISES

Factor:

1. $a^2 - x^2$.
2. $m^2 - r^2$.
3. $a^2 - 4$.
4. $x^2 - 9$.
5. $16 - x^2$.
6. $a^2 - 16\,b^2$.
7. $r^2 - 81$.
8. $1 - 25\,c^2$.
9. $9\,x^2y^2 - 1$.
10. $16\,x^2 - 25\,y^2$.
11. $25\,a^2 - 36\,b^2c^2$.
12. $4\,m^2 - 121$.
13. $9\,a^2 - 169\,x^2$.
14. $25\,r^2s^2 - 144\,x^6$.
15. $a^4 - 25\,b^4$.
16. $x^4y^4 - 9$.
17. $x^2y^8 - 64\,z^4$.
18. $9\,s^6 - 16\,t^{10}$.

19. $a^4 - 16$.

Solution. $a^4 - 16 = (a^2 - 4)(a^2 + 4)$
$= (a - 2)(a + 2)(a^2 + 4)$.

20. $a^4 - b^4$.
21. $x^4 - 1$.
22. $r^4 - 81$.
23. $16 - a^4$.
24. $81 - r^4$.
25. $x^{2m} - y^{2r}$.
26. $a^{2m} - b^{2r}$.
27. $c^{4a} - d^{4s}$.

EXERCISES

Factor:

1. $a^4 - x^8$.
2. $x^4y^4 - z^4$.
3. $(a - 2)^2 - c^2$.
4. $4(x + 3)^2 - y^2$.

5. $16(x-y)^2-z^2$.
6. $9(x-3y)^2-16r^4$.
7. $a^2-(b+c)^2$.
8. $4r^2s^2-(r-s)^4$.
9. $9a^2-4(b+3)^4$.
10. $(a-c)^2-(d+e)^2$.
11. $a^2(r+2s)^2-(x-y)^2$.
12. $x^2(2h-k)^2-(m-2r)^4$.
13. $4(a-x)^2-9a^2(c-2d)^4$.
14. $(x-y)^2-b^2x^2(3c-d)^2$.
15. $r^2+2rs+s^2-(x^2-2xy+y^2)$.
16. $x^2-y^2+2yz-z^2$.
17. $16h^2-25k^2+70kl-49l^2$.
18. $a^4-b^4-c^4-2b^2c^2$.
19. $x^2+6x+9-y^2+2yz-z^2$.
20. $4r^2+9s^2-16t^2-25m^4-12rs+40tm^2$.
21. $1-4x^2+4x^4-y^2-4yz-4z^2$.
22. $r^2s^2+2rs+1-c^2+10cd-25d^2$.
23. $121x^8-1-18y-81y^2$.
24. $a^2-b^2-(a-b)$.
25. $x^2-y^2+(ax+ay)$.
26. $m+n-m^2+n^2$.
27. x^2-4y^2+x-2y.
28. $r^2s^2-4s^2-m^2r^2+4m^2$.
29. $m^4x^4-m^4-81x^4+81$.
30. $x^{2n}-y^{6m}$.
31. $x^{4n+4}-r^{6m+8}$.
32. $.04m^2-.09r^2$.
33. $1.44x^4-1.21y^6$.
34. $\frac{1}{9}a^4-\frac{9}{16}y^2$.
35. $.04m^4-\frac{1}{81}b^{10}$.
36. $.0009r^4-.0025s^2$.
37. $\frac{36}{121}s^8-\frac{9}{25}r^4$.
38. $1.69a^2b^4-1.96c^2$.
39. $\frac{1}{16}s^6-\frac{1}{49}r^4$.

30. The quadratic trinomial. The type form is

$$x^2 + bx + c.$$

Since $(x + h)(x + k) = x^2 + (h + k)x + hk,$
it follows that $x^2 + bx + c$ can be factored into the binomial factors $(x + h)(x + k)$ if two numbers h and k can be found which have the sum b and the product c. The method of finding these factors is illustrated in the

EXAMPLE

Factor $x^2 + 2x - 35$.

Solution. Here $-35 = 1 \cdot -35$ or $-1 \cdot 35$ or $+7 \cdot -5$ or $+5 \cdot -7$. Of these pairs of factors of -35 only the pair $+7$ and -5 give the sum $+2$.

Hence $x^2 + 2x - 35 = (x + 7)(x - 5)$.

For factoring expressions of the type $x^2 + bx + c$ we have the

RULE. *Find two numbers whose algebraic product is* + c *and whose algebraic sum is* + b.

Write for the factors two binomials which have x *for their first terms and these numbers for the second terms.*

ORAL EXERCISES

Separate into binomial factors:

1. $x^2 + 5x + 6$.
2. $x^2 + 7x + 12$.
3. $x^2 + 7x + 6$.
4. $d^2 - 3d - 10$.
5. $c^2 - 2c - 35$.
6. $c^2 - 4c - 12$.
7. $s^2 - 10s + 16$.
8. $s^2 + 6s - 16$.
9. $s^2 - 5s - 14$.
10. $s^2 - 9s + 14$.

EXERCISES

Factor:

1. $r^2 - r - 90$.
2. $r^2 - 3r - 18$.
3. $x^2 - 2x - 24$.
4. $9 - 10x + x^2$.
5. $r^2 - 4rs + 3s^2$.
6. $m^2 + 8m - 20$.
7. $k^2 + 19k - 20$.
8. $k^2 - 21k + 20$.
9. $m^2 - 7mn + 10n^2$.
10. $1 - 5x + 6x^2$.
11. $1 + 2n - 24n^2$.
12. $m^2 - 6mr + 8r^2$.
13. $r^2 + 10rs + 16s^2$.
14. $1 + 3rs - 10r^2s^2$.
15. $r^2 - 6rs + 9s^2$.
16. $(a + b)^2 - 2(a + b) - 8$.
17. $(x - y)^2 + 4(x - y) + 3$.
18. $x^4 - 8x^2y^2 + 16y^4$.

Solution. $x^4 - 8x^2y^2 + 16y^4 = (x^2 - 4y^2)^2$
$= (x + 2y)^2(x - 2y)^2$.

19. $x^4 - 18x^2y^2 + 81y^4$.
20. $x^8 - 17x^4 + 16$.
21. $a^{2n} + 12a^n + 35$.
22. $a^{4x} - 5a^{2x} - 6$.
23. $b^{4y} - 2b^{2y} - 35$.
24. $s^{4x} - 10s^{2x} + 9$.

31. The general quadratic trinomial. The type form is

$$ax^2 + bx + c.$$

For many trinomials of this type two binomial factors of the form $(hx + k)(mx + n)$ may be found. The method of factoring such trinomials is illustrated in the

EXAMPLE

Factor $3x^2 + 7x - 6$.

Solution. $3x^2 + 7x - 6 = (?x + ?)(?x + ?)$.

To find the proper factors we must supply such numbers for the interrogation points in (1) and (2) as will give

$$\begin{array}{rl} ?x + ? & (1) \\ ?x + ? & (2) \\ \hline 3x^2 + ?x & \\ + ?x - 6 & \\ \hline 3x^2 + 7x - 6 & (3) \end{array}$$

$3x^2$ for the product of the first two terms of the binomials, -6 for the product of the last two terms of the binomials, $+7x$ for the sum of the cross products.

Now $3x^2 = 3x \cdot x$, (4)

and $-6 = -1 \cdot +6; 1 \cdot -6; +3 \cdot -2; -3 \cdot +2$. (5)

The factors of 3 and of -6 from (4) and (5) may be substituted for the interrogation points in (1) and (2) to form the following pairs of binomials, each having a product containing the first and last terms of the trinomial:

$$\begin{array}{cccccccc} 3x-1 & 3x+6 & 3x+1 & 3x-6 & 3x+3 & 3x-2 & 3x-3 & 3x+2 \\ x+6 & x-1 & x-6 & x+1 & x-2 & x+3 & x+2 & x-3 \end{array}$$

By trial we find that only the sixth pair has $+7x$ for the sum of its cross products, which gives the *middle* term of the trinomial.

Therefore $3x^2 + 7x - 6 = (3x - 2)(x + 3)$.

After a little practice it will usually be found unnecessary to write down all of the pairs of binomials that do not produce the required product.

If none of the pairs gives the required product, the given trinomial is *prime*.

If an expression of the form $ax^2 + bx + c$ is not prime, it can be factored by applying the

RULE. *Find two binomials, such that*

I. *The product of the first terms is* ax^2;

II. *The product of the last terms is* $+c$;

III. *The sum of the cross products is* $+bx$.

EXERCISES

Factor:

1. $2x^2 + 5x + 2$.
2. $2x^2 + 7x + 6$.
3. $2a^2 + 9a + 10$.
4. $4a^2 + 7a + 3$.
5. $3r^2 + 13r + 12$.
6. $3s^2 + 17s + 10$.
7. $3r^2 - 8r + 5$.
8. $6c^2 + 7cd + 2d^2$.
9. $4x^2 - 13xy + 10y^2$.
10. $10x^2 - 29x + 10$.
11. $12x^2 - 11xy + 2y^2$.
12. $2a^2 + 3a - 2$.
13. $3r^2 + r - 2$.
14. $2a^2 - a - 15$.
15. $12r^2 + 10r - 12$.
16. $12m^2 - 25mn + 12n^2$.
17. $12r^2 + 45rs - 12s^2$.
18. $12m^2 - 45mn - 12n^2$.
19. $3 - r - 10r^2$.
20. $10r^2 - 19rs - 15s^2$.
21. $6a^4 - 11a^3b - 7a^2b^2$.
22. $6x^3 + 10x^2y - 4xy^2$.
23. $2x^2 - 5xy - 3y^2$.
24. $2x^2 + 5xy - 12y^2$.
25. $6a^{2n} - 7a^n + 2$.
26. $3a^{4n} - 10a^{2n} - 8$.
27. $10x^{2n} - x^n - 3$.
28. $.72r^2 + 7rs + 12s^2$.
29. $\frac{2}{9}x^2 + \frac{1}{6}xy - \frac{1}{4}y^2$.
30. $.15r^2 - .24rs - .63s^2$.
31. $\frac{1}{2}r^2 - \frac{7}{12}rs + \frac{1}{6}s^2$.
32. $3r^{2m} - 8r^m s^{2n} + 5s^{4n}$.

32. Expressions reducible to the difference of two squares. The type form is

$$a^4 + ka^2b^2 + b^4.$$

If k has such a value that the trinomial is not a perfect square, a trinomial of this type can often be written as the *difference of two squares*. Thus, if $k = 1$, the addition and subtraction of a^2b^2 accomplishes this result.

Examples

1. Factor $a^4 + a^2b^2 + b^4$.

Solution. $a^4 + a^2b^2 + b^4 = a^4 + 2\,a^2b^2 + b^4 - a^2b^2$
$= (a^2 + b^2)^2 - (ab)^2$
$= (a^2 + b^2 + ab)(a^2 + b^2 - ab).$

2. Factor $25\,m^4 + 26\,m^2n^2 + 9\,n^4$.

Solution. If $4\,m^2n^2$ is added, the expression becomes a perfect trinomial square. Adding and subtracting $4\,m^2n^2$, we have

$25\,m^4+26\,m^2n^2+9\,n^4 = 25\,m^4+30\,m^2n^2+9\,n^4-4\,m^2n^2$
$= (5\,m^2 + 3\,n^2)^2 - (2\,mn)^2$
$= (5\,m^2+3\,n^2+2\,mn)(5\,m^2+3\,n^2-2\,mn).$

EXERCISES

Factor:

1. $x^4 + x^2y^2 + y^4$.
2. $r^4 + r^2s^2 + s^4$.
3. $a^4 + a^2b^4 + b^8$.
4. $r^4 + 3\,r^2s^2 + 4\,s^4$.
5. $m^8 + m^4 + 1$.
6. $x^4 - x^2y^2 + 16\,y^4$.
7. $x^4 - 12\,x^2y^2 + 16\,y^4$.
8. $25\,a^4 - 19\,a^2 + 1$.
9. $25\,x^4 - 11\,x^2 + 1$.
10. $16\,r^4 - 17\,r^2s^4 + s^8$.
11. $4\,r^4 - 44\,r^2s^2 + 49\,s^4$.
12. $25\,x^4 - 19\,x^2 + 9$.
13. $81\,a^4 + 11\,a^2b^2 + 4\,b^4$.
14. $36\,a^4 - 25\,a^2b^2 + 4\,b^4$.
15. $4\,a^4 + 1$.

HINT. $4\,a^4 + 1 = 4\,a^4 + 4\,a^2 + 1 - 4\,a^2$.

16. $c^4 + 4\,d^4$.
17. $64\,a^4x^2 + x^2$.
18. $4\,a^{4n} + b^{4n}$.
19. $x^{4a+4} + 4\,y^{8a+4}$.

33. A binomial the sum or the difference of two cubes. The type form is

$$a^3 \pm b^3.$$

Dividing $a^3 + b^3$ by $a + b$ gives the quotient $a^2 - ab + b^2$, and $a^3 - b^3$ divided by $a - b$ gives the quotient $a^2 + ab + b^2$.

Therefore $a^3 + b^3 = (a + b)(a^2 - ab + b^2)$, (1)

and $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$. (2)

Formulas (1) and (2) above may be applied as in the

Examples

1. Factor $h^3 + 8$.

Solution. $h^3 + 8 = h^3 + 2^3 = (h + 2)(h^2 - 2h + 2^2)$
$= (h + 2)(h^2 - 2h + 4)$.

2. Factor $27 - y^3$.

Solution. $27 - y^3 = 3^3 - y^3 = (3 - y)(3^2 + 3y + y^2)$
$= (3 - y)(9 + 3y + y^2)$.

EXERCISES

Factor:

1. $m^3 - 64$.
2. $c^3 + b^6$.
3. $r^3 - s^9$.
4. $(2a)^3 - (3b)^3$.
5. $8r^3 - 27s^3$.
6. $125x^3 + 8y^6$.
7. $8x^3y^6 + 1$.
8. $216 - r^3$.
9. $m^6 + n^6$.
10. $(a + b)^3 + c^3$.
11. $m^6 - r^6$.

Solution.

$m^6 - r^6 = (m^3 + r^3)(m^3 - r^3)$
$= (m + r)(m^2 - mr + r^2)(m - r)(m^2 + mr + r^2)$.

12. $x^6 - y^6$.
13. $a^6 - 64$.
14. $x^6 - 64y^6$.
15. $1 - a^6$.
16. $a^{12} - b^6$.
17. $x^{12} - 1$.

18. $a^3 + b^3 + a + b$.

Solution.
$$\begin{aligned} a^3 + b^3 + a + b &= (a^3 + b^3) + (a + b) \\ &= (a + b)(a^2 - ab + b^2) + (a + b) \\ &= (a + b)(a^2 - ab + b^2 + 1). \end{aligned}$$

19. $x^3 - y^3 + x - y$.

20. $m^3 - n^3 - m + n$.

21. $x^3 - 8y^3 + x - 2y$.

22. $a^3 - 8b^3 + a - 2b$.

23. $r^3 + 8(a + b)^3$.

24. $r^3 s^6 t^9 - 64 k^{15}$.

25. $x^3 - \frac{1}{r^3}$.

26. $(a + b)^3 + (a - b)^3$.

27. $a^{3n} + b^{3n}$.

28. $8x^{3n}y^m - 729y^{m+3}$.

29. $27m^{3r+3} + 64r^{6x+9}$.

34. The Remainder Theorem. If any rational integral expression in x be divided by $x - n$, the remainder is the same as the original expression with n substituted for x. This fact is illustrated in the

EXAMPLE

Divide $x^2 - 7x + 12$ by $x - n$.

Solution.
$$\begin{array}{l} x + (n - 7) \\ x - n \,\big|\, x^2 - 7x + 12 \\ \quad x^2 - nx \\ \hline \quad (n - 7)x + 12 \\ \quad (n - 7)x \qquad - n^2 + 7n \\ \hline \quad \text{Remainder} = n^2 - 7n + 12. \end{array}$$

Here the remainder $n^2 - 7n + 12$ is the same as $x^2 - 7x + 12$, the given expression, when n is substituted for x.

EXERCISES

1. Divide $x^2 + bx + c$ by $x - n$ and show that the remainder is $n^2 + bn + c$.

2. Divide $x^2 + bx + c$ by $x - a$ and find the remainder.

3. Divide $x^3 + ax^2 + bx + c$ by $x - n$ and find the remainder.

4. In $(x^3 + x^2 - 5x + 3) \div (x - 2)$, find the remainder ($a$) by division, ($b$) by the Remainder Theorem.

Solution. (b) $2^3 + 2^2 - 5 \cdot 2 + 3 = 5$.

5. In $(x^3 - x + 5) \div (x - 3)$, find the remainder ($a$) by division, ($b$) by the Remainder Theorem.

By use of the Remainder Theorem find the remainders in the following:

6. $(x^3 + x^2 - 5x + 8) \div (x - 3)$.

7. $(x^3 - 3x - 15) \div (x + 4)$.

8. $(r^3 - 2r^2 - 100) \div (r - 5)$.

9. $(s^3 - 2s^2 - 2s - 3) \div (s + 3)$.

10. $(r^4 - 2r^3 + r - 2) \div (r - 2)$.

11. $(r^4 - 3r^2 + 2r - 1) \div (r + 2)$; $\div (r - 1)$.

12. $(2m^4 - 4m^3 + m^2 - 4m + 4) \div (m - 1)$.

35. Factor Theorem. By substituting 3 for x in $x^2 - 7x + 12$ we obtain $9 - 21 + 12$, or 0. Thus the Remainder Theorem shows, without actually performing the division, that $x - 3$ will divide $x^2 - 7x + 12$ without a remainder; that is, divide it exactly. Again, if 4 is substituted for x in $x^2 - 7x + 12$, the expression equals zero. Hence $x - 4$ is a factor of $x^2 - 7x + 12$. These examples illustrate the

THEOREM. *If any rational integral expression in* x *becomes zero when a number* n *is substituted for* x, *then* x − n *is a factor of the expression.*

The Factor Theorem may be used to factor some of the preceding exercises and, in addition, many others which are very difficult to factor by previous methods.

NOTE. By means of the Factor Theorem we are able to solve cubic and higher equations when the roots are integers. The solution of the general cubic equation is one of the famous problems of mathematics and one which is accompanied by many interesting applications. This problem was first solved by the Italian, Tartaglia, about 1530, but was published by Cardan, to whom Tartaglia explained his solution on the pledge that he would not divulge it. For many years the credit for the discovery was given to Cardan, and to this day it is usually called Cardan's Solution.

When searching for the values of x which will make an expression zero, only integral divisors of the last term of the expression (arranged according to the descending powers of x) need be tried, for the last term of the factor must be an integral divisor of the last term of the expression.

EXAMPLE

Factor $x^3 + 4x + 5$.

Solution. If $x - n$ is a factor of $x^3 + 4x + 5$, then n must be an integral divisor of 5. Now the factors of $+5$ are $1, -1, 5$, and -5. If -1 is put for x, then $x^3 + 4x + 5$ equals zero, hence $x + 1$ is a factor of $x^3 + 4x + 5$. Dividing $x^3 + 4x + 5$ by $x + 1$, we obtain the quotient $x^2 - x + 5$. Since $x^2 - x + 5$ is prime, the factors of $x^3 + 4x + 5$ are $x + 1$ and $x^2 - x + 5$.

ORAL EXERCISES

1. Is $(x - 1)$ a factor of $x^3 + 3x - 4$?
2. Is $(x - 2)$ a factor of $2x^3 + x^2 - 20$?
3. Is $(a - 2)$ a factor of $a^3 - 3a + 2$?
4. Is $(x - 1)$ a factor of $x^3 + 3x^2 - 4$?

5. Is $(r + 1)$ a factor of $r^3 - 4r^2 - 4r + 1$?

6. Is $(r - 3)$ a factor of $2r^3 - r^2 + 5$?

7. Is $(s + 1)$ a factor of $3s^3 - 5s^2 + 8$?

8. Is $(m - 3)$ a factor of $2m^3 - 5m^2 - 9$?

EXERCISES

Factor:

1. $x^3 + x - 2$.
2. $x^3 + 2x + 3$.
3. $a^3 + a^2 - 36$.
4. $x^3 + x - 10$.
5. $r^3 + r^2 - 12$.
6. $s^3 - 2s^2 - 5s + 6$.
7. $r^3 - r^2 + 4r - 4$.
8. $x^3 - x^2 - 4$.
9. $y^3 - y^2 - 9y + 9$.
10. $x^3 + 2x^2 + 3x - 6$.
11. $r^3 - 7r + 6$.
12. $r^3 - 4r^2 + 4r - 16$.
13. $x^4 - 7x^2 - 6x$.
14. $x^3 - 7x^2 + 4x + 12$.
15. $2x^3 - 2x^2 - x - 6$.
16. $x^3 - x^2 - 48$.

36. The sum or difference of two like powers. The type form is

$$a^n \pm b^n.$$

The cases in which $a^n \pm b^n$ is divisible by $a + b$ or $a - b$ can be determined by the Factor Theorem.

Thus in $a^n - b^n$, where n is either an odd or an even integer, substitute b for a. Then $a^n - b^n$ becomes $b^n - b^n = 0$. Therefore $a - b$ is always a factor of $a^n - b^n$.

In $a^n - b^n$, n being even, put $-b$ for a. Then $a^n - b^n$ becomes $b^n - b^n = 0$, since $(-b)^n$ is *positive* when n is *even*. Therefore, when n is even, $a + b$ as well as $a - b$ is an exact divisor of $a^n - b^n$.

In $a^n + b^n$, n being even, put either $+b$ or $-b$ for a. Then $a^n + b^n$ becomes $b^n + b^n$, which is not zero. There-

fore $a^n + b^n$ is never divisible by $a + b$ or $a - b$ when n is even.

In $a^n + b^n$, n being odd, put $-b$ for a. Then $a^n + b^n$ becomes $(-b)^n + b^n = 0$, since $(-b)^n$ is *negative* when n is *odd*. Therefore when n is odd $a + b$ is a divisor of $a^n + b^n$.

Summing up:

I. $a^n - b^n$ is *always* divisible by $a - b$.

II. $a^n - b^n$, when n is *even*, is divisible both by $a + b$ and by $a - b$.

III. $a^n + b^n$ is *never* divisible by $a - b$.

IV. $a^n + b^n$, when n is *odd*, is divisible by $a + b$.

ORAL EXERCISES

State a binomial factor for each of the following:

1. $x^3 - y^3$.	6. $a^8 - b^8$.	11. $1 + r^7$.
2. $r^3 - 5^3$.	7. $a^{10} - b^{10}$.	12. $1 + x^5$.
3. $27 - b^3$.	8. $m^3 + 2^3$.	13. $3^3 + 1$.
4. $m^5 - 2^5$.	9. $a^3 + 8$.	14. $4^3 - 1$.
5. $32x^5 - r^5$.	10. $r^5 + y^5$.	15. $10^3 - 1$.

16. Is $10^5 + 1$ divisible by 11?

17. Is $10^9 - 1$ divisible by 9?

EXAMPLE

Factor $a^5 + b^5$.

Solution. By division,

$$\frac{a^5 + b^5}{a + b} = a^4 - a^3b + a^2b^2 - ab^3 + b^4.$$

Hence $a^5 + b^5 = (a + b)(a^4 - a^3b + a^2b^2 - ab^3 + b^4)$.

Note that the signs of the second factor are alternately plus and minus. Also note the order in which the exponents occur.

EXERCISES

Factor:

1. $x^5 + z^5$.
2. $r^5 + 1$.
3. $a^5 + 2^5$.
4. $x^5 + 32$.
5. $(a^2)^5 + (b^3)^5$.
6. $a^5 - b^5$.
7. $a^5 - 2^5$.
8. $a^5 - 32\,x^5$.
9. $a^7 - x^7$.
10. $1 - r^7$.
11. $x^7 - 128$.
12. $x^7 + 128\,a^7$.
13. $r^{10} + 32\,x^{10}$.
14. $x^7 + a^7$.
15. $1 + b^7$.
16. $x^5y^5 - 32$.
17. $r^7y^7 - 128$.
18. $a^{15} + 1$.
19. $1 - r^n$.

HINT. Write only the first five terms and the last term of the polynomial factor.

20. $1 + r^n$.
21. $x^6 - y^6$.
22. $x^8 - y^8$.
23. $a^{10} - b^{10}$.
24. $a^{12} - b^{12}$.
25. $(a + b)^5 - b^5$.

37. General directions for factoring. The following suggestions will prove helpful in factoring:

I. *First look for a common monomial factor, and if there is one (other than 1), separate the expression into its greatest monomial factor and the corresponding polynomial factor.*

II. *Then from the form of the polynomial factor determine with which of the following types it should be classed, and use the methods of factoring applicable to that type.*

1. $ax + ay + bx + by$.
2. $a^2 \pm 2\,ab + b^2$.
3. $a^2 - b^2$.
4. $x^2 + bx + c$.
5. $ax^2 + bx + c$.
6. $a^4 + ka^2b^2 + b^4$.
7. $a^3 \pm b^3$.
8. $a^n \pm b^n$.

III. *Proceed again as in II with each polynomial factor obtained, until the original expression has been separated into its prime factors.*

IV. *If the preceding steps fail, try the Factor Theorem.*

REVIEW EXERCISES

Factor:

1. $3r^2 + 6r^3 + 9r^4$.
2. $20bc + 10bd - 15be$.
3. $3\pi r^2 + 3\pi rh$.
4. $r^3 + 6r^2y + 9ry^2$.
5. $r^4 + 4br^3 + 2b^2r^2$.
6. $r^4 - 2a^2r^2$.
7. $6x^4 + 2x^3 + 2x^2$.
8. $5a^3 + 2a^2 - 15a - 6$.
9. $a^2 + 4ab + 4b^2$.
10. $2c^3d - 8cd^3$.
11. $3m^3 - 3m^2 - 18m$.
12. $40a^3 - 5$.
13. $r^4 - 2br^3 + 4b^2r^2$.
14. $r^4 - 8a^3r$.
15. $(b + c)y + bz + cz$.
16. $r^3 - .04r$.
17. $r^3 - 4r(b + 2c)^2$.
18. $2x^2 + 3ax + a^2$.
19. $x^4 - 7x^2y^2 + 9y^4$.
20. $a^4c - ac^4$.
21. $2x^7y - 2xy^7$.
22. $x^5 - 2x^4 - 9x^2$.
23. $\frac{3}{2}rx + \frac{3}{2}ry - \frac{3}{2}rz$.
24. $m^3 - 24m^2 + 144m$.
25. $ac + 2bc - ad - 2bd$.
26. $18r^3 - 22r^2s + 8rs^3$.
27. $.04x^4 - .81y^2$.
28. $4b^5 + 5b^4c - 6b^3c^2$.
29. $x^3 + 4x - 5$.
30. $ax^2 - 4a + 3x^2 - 12$.
31. $x - x^2 - x^3 + x^4$.
32. $m^2 - n^2 - 2np - p^2$.
33. $r^2 + ar - ab - br$.
34. $b^2 - 2bc + c^2 - d^2$.
35. $x^3 - 8x^2 - x + 8$.
36. $a^3 + 27a^2$.
37. $64d^3 + 2d^8$.
38. $2x^3 - 2x^2 - 12x$.
39. $3b^7 - 3b$.
40. $1 - 5x^4 + 4x^8$.
41. $16m^4 - .0081$.
42. $5br^9 - 5b$.

43. $18\,r^3 - 3\,r^2 - 36\,r$.

44. $7\,m - 7\,m^3n^4$.

45. $(r^2 + s^2)^4 - 16\,r^4s^4$.

46. $(r^2 - 6)^2 - r^2$.

47. $16\,x^2y^2 - (x - y)^4$.

48. $a^4 + ab^{15}$.

49. $a^5 + x^{15}$.

50. $6\,x^2 - 13\,x + 6$.

51. $x^4 - x^2y^2 + x^3 - x^2y$.

52. $a^2 - 12\,a + 36 - a^4$.

53. $a^2 - b^2 + (a - b)^2$.

54. $m^4 - 7\,m^2n^2 + n^4$.

55. $32\,x^8 - x^3y^{10}$.

56. $5(a - b)^2 - a + b$.

57. $c^2 + 4\,d^2 - x^2 - 4\,cd$.

58. $x^2 - 20 + x^4$.

59. $6\,a^4 + 3\,a^3 - 3\,a^2$.

60. $16\,a^4 + 7\,a^2 + 1$.

61. $x^6y^6 - 64$.

62. $x^7 - 18\,x^5 - 27\,x^4$.

63. $a^7 - a^2b^5 + a^2b - a^3$.

64. $x^2 - 5(2\,x - 5)$.

65. $4\,r^2 + 23\,r + 15$.

66. $a^4x^4 + 4$.

67. $x^8y^6 + x^2z^6$.

68. $x^3 - 10\,x - 3$.

69. $a^8 - 5\,a^4 + 4$.

70. $a^4 - 9\,a^2 - a + 3$.

71. $a^9 - 64\,a^3 - a^6 + 64$.

72. $4\,ab + (a - b)^2$.

73. $x - 1 + x^5 - x^3$.

74. $x^3 + x - y - y^3$.

75. $1 - abc^3 + (b - a^2)c^2$.

76. $2\,x^3 + 3\,x^2 + x$.

77. $m^7 - 8\,m - 7\,m^4$.

78. $3\,a^3 - 9\,a(2\,a - 3)$.

79. $2\,x^4 - 10\,x^2 + 4\,x$.

80. $10\,a - 7\,a^2 - 6\,a^3$.

81. $a^nx^2 + 2\,a^nx + a^n$.

82. $h^{2m} - 2\,h^mk^n + k^{2n}$.

83. $a^{2m} - b^{2n}$.

84. $a^{4m} + 2\,a^{2m}b^{2n} + b^{4n}$.

85. $a^{3m} + b^{3n}$.

86. $a^{3m} - b^{3n}$.

38. Solution of linear equations by factoring. Many linear equations in one unknown can be solved through the application of factoring and division to the members of the equation, if one member has been made to contain the terms which involve the unknowns.

EXERCISES

Solve Exercises 1–9 for x:

1. $xm + xn = km + kn$.
2. $cx + dx = c^2 + cd$.
3. $ax - cx + bc = ab$.
4. $5\,ax - 3\,bx - 5\,a^2 + 3\,ab = 0$.
5. $ax = 3\,ad + bx - 3\,bd$.
6. $cx - 2\,dx = c^2 - 4\,d^2$.
7. $ax - x + 1 - a = 0$.
8. $bx + 3\,dx - b^2 + 9\,d^2 = 0$.
9. $x(2\,a - 7\,c) = 4\,a^2 - 28\,ac + 49\,c^2$.
10. Solve for m, $m(3\,a - c) - 9\,ad = 2\,ec - 6\,ae - 3\,cd$.
11. Solve for x, $ax - 3\,bx - a^2 = 3\,b^2 - 4\,ab$.
12. Solve for s, $2\,as - 7\,s + 13\,a = 2\,a^2 + 21$.
13. Solve for t, $2\,te - 2\,e + 15 - 8\,e^2 + 3\,t = 0$.
14. Solve for y, $y + 21\,d^2 + 4\,d = 7\,dy + 1$.
15. Solve for m, $am + bm - cm = a^2 + b^2 - c^2 + 2\,ab$.
16. Solve for x, $ax - a^3 = bx - b^3$.
17. Solve for h, $ah - bh - a^2 = ch - 2\,bc - c^2 - b^2$.

39. Solution of equations by factoring. The methods of factoring enable us to solve many equations in one unknown. In the solution of equations by factoring use is made of the

PRINCIPLE. ***If the product of two or more factors is zero, at least one of the factors must be equal to zero.***

Each of the factors *may* be zero, but the vanishing of one factor is *sufficient* to make the product zero. This is illustrated in the following example.

Example

Solve the equation $x^3 - 3x^2 = 10x$.

Solution. Transposing $10x$, we have

$$x^3 - 3x^2 - 10x = 0.$$

Factoring, $$x(x + 2)(x - 5) = 0.$$

Solving the three equations $x = 0$, $x + 2 = 0$, and $x - 5 = 0$, which result from setting each factor separately equal to zero, we have $x = 0, -2$, and $+5$.

Substituting these values in the original equation, we find that the three numbers $0, -2$, and $+5$ are roots of that equation.

The method of solving the above example is stated in the

Rule. *If necessary, rewrite the equation so that the second member is zero. Then factor the first member, set each factor which contains an unknown equal to zero, and solve the resulting equations.*

Check as usual.

In the above example the division of each member of $x^3 - 3x^2 = 10x$ by x gives $x^2 - 3x = 10$. This last equation has only the roots -2 and 5, while the original equation had, in addition, the root 0. That is, one root has been lost sight of by dividing the equation by the factor x.

Again, if the equation $x^2 + 2x - 8 = x - 2$ is solved by the rule, the roots are -3 and 2. If, however, each member of $x^2 + 2x - 8 = x - 2$ is divided by $x - 2$, the resulting equation $x + 4 = 1$ has the root -3 only. In each of these cases the root would not have been lost if the factor used as a divisor had been set equal to zero and the equation thus obtained had been solved.

EXERCISES

Solve by factoring:

1. $x^2 - 4 = 0$.
2. $x^2 - 5x = 0$.
3. $x^2 - ax = 0$.
4. $x^2 - 4a^2 = 0$.
5. $4r^3 = 25r$.
6. $a^3 = 64a$.
7. $r^2 + 5 = 6r$.
8. $2a^2 + a = 6$.
9. $x^2 - xs - 2s^2 = 0$.
10. $x^2 - 2ax - 8a^2 = 0$.
11. $y^3 - y + 2 = 2y^2$.
12. $x^3 - ax^2 - 12a^2x = 0$.
13. $ar + 3r = a^2 - 9$.
14. $cx - c^2 + d^2 = dx$.
15. $5r^2 - 4r = 1$.
16. $2r^2 - 3r - 35 = 0$.
17. $24r^2 - 49r + 15 = 0$.
18. $x^3 + x^2 - a^2x - ax = 0$.
19. $4y + 19y^2 - 5y^3 = 0$.
20. $y^3 - 7y - 6 = 0$.
21. $y^4 - 13y^2 + 36 = 0$.
22. $x^3 - 6x^2 - 16x = -96$.
23. $x^5 - 5x^3 = -4x$.
24. $x^4 - 7x^2 = -6x$.
25. $x^3 - x^2 - x + 1 = 0$.
26. $x^3 - x^2 - 24x - 36 = 0$.
27. $x^4 - 5x^2 + 4 = 0$.
28. $r^4 - 26r^2 = -25$.
29. $3r^3 + 24r^2 + 48r = 0$.
30. $x^3 - x^2 - 14x + 24 = 0$.
31. $x^4 - 65x^2 + 64 = 0$.
32. $r^3 + 3r^2 + 3r + 1 = 0$.
33. $r^2 - 7r - 8 = r + 1$.
34. $4h^2 + h - 1 = 2h - 1$.

40. The highest common factor. *The highest common factor* (H. C. F.) of two or more monomials or polynomials is the expression of highest degree, with the greatest numerical coefficient, which is an exact divisor of each.

Thus the H. C. F. of $18\,r^2s^3$ and $27\,rs^4$ is $9\,rs^3$, and the H. C. F. of $x^2 - 4\,x + 3$ and $x^2 - 9$ is $x - 3$.

EXAMPLE

Find the H. C. F. of $4\,a^5 - 28\,a^4 + 48\,a^3$ and $8\,a^3 + 8\,a^2 - 96\,a$.

Solution. Factoring, we have

$$4\,a^5 - 28\,a^4 + 48\,a^3 = 4\,a^3(a - 3)(a - 4).$$
$$8\,a^3 + 8\,a^2 - 96\,a = 8\,a(a - 3)(a + 4).$$

Therefore, the H. C. F. is $2^2a(a - 3)$ or $4\,a^2 - 12\,a$.

The method used in the preceding solutions for finding the H. C. F. of two or more monomials or polynomials is stated in the

RULE. ***Separate each expression into its prime factors. Then find the product of such factors as occur in each expression, using each prime factor the least number of times it occurs in any one expression.***

If two or more polynomials have no common factor other than 1, then 1 is their H. C. F., and the polynomials are said to be prime to each other.

EXERCISES

Find the H. C. F. of the following:

1. 9, 12, 15.
2. 12, 18, 24.
3. 24, 60, 72.
4. 30, 45, 90.

5. 60, 120, 144.

6. $12\,a^2$, $30\,a^4$, $36\,a^5$.

7. $30\,c^2d^3$, $45\,cd^2$, $75\,c^2d$.

8. $28\,a^2b^4c^3$, $42\,ab^5c$, $70\,a^3bc^2$.

9. $66\,ac^4x$, $132\,a^3c^2x^3$, $165\,a^2c^4x^2$.

10. $a^2 + 2\,ab + b^2$, $a^2 - b^2$.

11. $3\,a^2 - 3\,b^2$, $9(a - b)^2$, $3\,a^3 - 3\,b^3$.

12. $ax^2 - 2\,axy + ay^2$, $a^2x^2 - a^2y^2$, $2\,ax^3 - 2\,ay^3$.

13. $3\,x^4 - 6\,x^3y$, $6\,x^6 - 24\,x^4y^2$, $12\,x^5 - 96\,x^2y^3$.

14. $5\,x^7 - 160\,x^2$, $15\,x^5 - 60\,x^3$, $25\,x^7 - 200\,x^4$.

15. $4\,x^4y^4 - 4\,x^2y^6$, $5\,x^7y^4 - 5\,x^3y^8$, $8\,x^{11}y^4 - 8\,x^5y^{10}$.

16. $a^{2n}b^5 - a^nd^4$, $a^{3n}c^5 - 3\,a^{2n}$, $4\,a^{2n} - a^{2n}d^3$.

17. $a^{2n} - 2\,a^nb + b^2$, $a^{2n} - b^2$, $a^{3n} - b^3$.

NOTE. The most famous, and in some respects the most perfect, treatise on elementary mathematics ever written is Euclid's "Elements." About one third of the material of the thirteen books treats topics which to-day would be considered arithmetical in character. In appearance and language, however, they are all geometrical, for Euclid represents quantities not by numerals, as we do in arithmetic, or by letters, as we do in algebra, but by lines. Book VII contains the earliest statement of a general method for finding the Highest Common Factor of two numbers. This method, though never necessary in elementary mathematical work, is of fundamental importance in advanced portions of algebra. It is so perfect and beautiful from a scientific point of view that until recently it remained in elementary treatises on algebra and arithmetic by force of tradition. It is a great tribute to Euclid's genius that he was able to devise so perfect a method for the process that all the efforts of two thousand years have been unable to improve it essentially.

CHAPTER IV

FRACTIONS

41. Operations on fractions. The changing of a fraction to higher or to lower terms, and the addition and the subtraction of fractions in both arithmetic and algebra, depend on the

PRINCIPLE. *The numerator and the denominator of a fraction may be multiplied by the same expression or divided by the same expression without changing the value of the fraction.*

Thus $\frac{2}{3} = \frac{2 \cdot 4}{3 \cdot 4} = \frac{8}{12}$ and $\frac{24}{32} = \frac{24 \div 8}{32 \div 8} = \frac{3}{4}$.

Similarly, $\frac{n}{d} = \frac{n \cdot x}{d \cdot x} = \frac{nx}{dx}$ and $\frac{n}{d} = \frac{n \div x}{d \div x} = \frac{\frac{n}{x}}{\frac{d}{x}}$.

It should be noted that by the application of this principle a fraction is changed in form but not in value.

ORAL EXERCISES

Divide both numerator and denominator of

1. $\frac{10}{25}$ by 5.
2. $\frac{12}{30}$ by 6.
3. $\frac{6x}{15}$ by 3.
4. $\frac{2x}{3x^2}$ by x.
5. $\frac{12x}{8x^3}$ by $4x$.
6. $\frac{15a^3}{12a}$ by $3a$.

7. $\frac{12\,x^2y^3}{15\,xy^4}$ by $3\,xy^3$.

8. $\frac{18\,x^2yz^3}{24\,xy^2z}$ by $6\,xyz$.

9. $\frac{x+2}{x^2-4}$ by $x+2$.

10. $\frac{3\,x-6}{x^2-4}$ by $x-2$.

Reduce to lowest terms:

11. $\frac{2\,ax}{3\,a}$.

12. $\frac{5\,x^2y}{10\,xy}$.

13. $\frac{2(x-5)}{3\,x(x-5)^2}$.

14. $\frac{3(a-2)}{a^2-4}$.

15. $\frac{5\,x+10}{2\,x+4}$.

16. $\frac{x^3+27}{x(x+3)}$.

17. $\frac{x^2+4}{x^4-16}$.

18. $\frac{x^3-8}{x^2-4\,x+4}$.

EXERCISES

Reduce to lowest terms:

1. $\frac{2\,a^2b^2}{10\,a^2b^3}$.

2. $\frac{15\,xy^3}{245\,x^3y}$.

3. $\frac{169\,s^3t^4z^2}{65\,s^4t^3z^3}$.

4. $\frac{x^2-9}{2\,x^2+6\,x}$.

5. $\frac{5\,x^2+5\,xy}{25\,x^4+25\,x^2y^2}$.

6. $\frac{r^2-3\,r-4}{r^2+5\,r+4}$.

7. $\frac{s^2-1}{s^2+3\,s+2}$.

8. $\frac{2\,a^3+8\,a^2b+8\,ab^2}{5\,a^5-20\,a^3b^2}$.

9. $\frac{48\,x^4-6\,xy^3}{32\,x^6-32\,x^5y+8\,x^4y^2}$.

10. $\frac{3\,x^3-375}{2\,x^2+10\,x+50}$.

11. $\frac{a^3+6\,a^2b+9\,ab^2}{3\,a^2b+9\,ab^2}$.

12. $\frac{2\,c^2+2\,cd-4\,d^2}{5\,c^2-5\,d^2}$.

13. $\frac{r^3+r^2-5\,r-6}{r^3+8}$.

14. $\frac{a^3+b^3}{a^4+a^2b^2+b^4}$.

15. $\dfrac{r^5 - s^5}{r^2 - 4rs + 3s^2}$.

16. $\dfrac{m^3 + n^3}{m^5 + n^5}$.

17. $\dfrac{r^6 - 64}{r^6 + 4r^3 - 32}$.

18. $\dfrac{3x^3 - 9x^2 - 30x}{2x^3 + 8x^2 + 8x}$.

19. $\dfrac{5x^4 - 40x}{3x^5 - 96}$.

20. $\dfrac{12a^3 + 10a^2 - 12a}{4 + 9a^2 - 12a}$.

21. $\dfrac{r^3 - 7r + 6}{r^3 - 8 + 14r - 7r^2}$.

22. $\dfrac{ab - ac}{ab + 3b - 3c - ac}$.

23. $\dfrac{3h^2 + 16h - 35}{5h^2 + 33h - 14}$.

24. $\dfrac{(\frac{2}{3})^2 - 1}{\frac{2}{3} + 1}$.

25. $\dfrac{(\frac{3}{5})^3 - 1}{\frac{3}{5} - 1}$.

26. $\dfrac{(\frac{3}{2})^2 - (\frac{3}{2})(\frac{2}{5}) + (\frac{2}{5})^2}{(\frac{3}{2})^3 + (\frac{2}{5})^3}$.

27. $\dfrac{1 - r^2}{(1 + rs)^2 - (r + s)^2}$.

28. $\dfrac{r^2 + r + s - s^2}{1 - (r - s)^2}$.

42. Changes of sign in a fraction. In the various operations on fractions three signs must be considered: the sign of the numerator, the sign of the denominator, and the sign before the fraction. Since a fraction is an indicated quotient, the law of signs in division, when considered in connection with the sign of a fraction, gives the following identity:

$$+\frac{+n}{+d} = +\frac{-n}{-d} = -\frac{-n}{+d} = -\frac{+n}{-d}$$

From the above we have the

PRINCIPLE. *In a fraction the signs of both numerator and denominator, or the sign of the numerator and the sign before the fraction, or the sign of the denominator and the sign before the fraction, may be changed without altering the value of the fraction.*

In applying this principle to a polynomial denominator or a numerator, like $2n^2 - 3n - 5$, we change its sign

by changing the signs of every term of the polynomial. If the polynomial denominator is in factored form like $(t - 3)(2t - 1)(3t + 1)$, its sign is changed by changing the sign of *any* one factor or of an *odd number* of factors.

ORAL EXERCISES

Express the following as equivalent fractions in three additional ways, using the above principle:

1. $\frac{-a}{b}$. 2. $\frac{3}{-x}$. 3. $\frac{r}{2-t}$. 4. $-\frac{2n}{1-n}$.

Without changing their values, change the signs in the following indicated products:

5. $(s - 2)(s + 2)$.
6. $(5x + 2)(3x - 5)$.
7. $(x - a)(x - b)$.
8. $-(x + y)(x - y)$.
9. $(x + ab)(ab - x)$.
10. $-x(a - b)(c - d)$.
11. $(x + 1)(x - 1)(x^2 + 1)$.
12. $(x - a)(x - b)(c - x)$.

13. Find the indicated product in Exercise 12. Then change the sign of two factors and again find the product. Compare the results.

14. Make a general statement of which the result of Exercise 13 is an illustration.

State equivalent fractions for the following, each of which shall contain the factor $x - 1$ in its denominator:

15. $\frac{1}{1-x}$. 16. $-\frac{2}{1-x^2}$. 17. $-\frac{-5}{1-x^3}$.

18. $\frac{s-2}{x(1-x)^3}$. 20. $\frac{-3d-5}{-1+2x-x^2}$.

19. $\frac{5a}{2-x-x^2}$. 21. $-\frac{t^2-t+1}{1-x^3}$.

By changes of sign make the denominators of the following fractions alike, without changing the values of the fractions:

22. $\dfrac{3}{(a-2)(a+2)}, \dfrac{a+3}{(2-a)(2+a)}$.

23. $\dfrac{m}{(m-n)(p-m)(n-p)}, \dfrac{n}{(n-m)(p-m)(n-p)}, \dfrac{p}{(m-n)(m-p)(p-n)}$.

43. Lowest common multiple. The lowest common multiple (L. C. M.) of two or more arithmetic or algebraic expressions is the expression with the least number of factors which will exactly contain each of the given expressions.

Thus the L. C. M. of 6, 9, and 12 is 36, and the L. C. M. of m^2n, $2\,mn^2$, and $3\,mn^3r^2$ is $6\,m^2n^3r^2$.

EXAMPLE

Find the L. C. M. of $2\,n^3 + 2\,n^2$, $6\,n^5 - 6\,n^3$, and $9\,n^3 - 18\,n^2 + 9\,n$.

Solution. Factoring, $2\,n^3 + 2\,n^2 = 2\,n^2(n+1)$

$6\,n^5 - 6\,n^3 = 6\,n^3(n+1)(n-1)$

$9\,n^3 - 18\,n^2 + 9\,n = 9\,n(n-1)^2$

Therefore the L. C. M. $= 18\,n^3(n+1)(n-1)^2$.

For such expressions as can be readily factored the method of finding the L. C. M. is stated in the

RULE. *Separate each expression into its prime factors. Then find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one expression.*

ORAL EXERCISES

Find the L. C. M. of the following:

1. 4, 10, 12.
2. 25, 75, 100.
3. 42, 56, 21.
4. $4x$, $6x^2$, $8x$.
5. $2ab^2$, $8a^2b^2$, $6ab^3$.
6. $5a^2b^2$, $20a^3b$, $25a^2b^3c$.
7. $7a^3b^4$, $21a^2b^2$, $35abc^4$.
8. $7x^4y^6$, $8x^3y^3$, $6x^5y^4$.
9. $2(x+2)$, x^2-4, $2(x-2)$.
10. x^3-8, $2(x-2)$, $2x^2+4x+8$.
11. x^2-9, x^2+6x+9.
12. $x-5$, x^2-x-20.

EXERCISES

Find the L. C. M. of the following:

1. a^3+ab^2, $a^2-2ab+b^2$, a^2-ab.
2. $ax-ay+bx-by$, a^2-b^2, x^2-y^2.
3. $c^2-6cd+9d^2$, c^2-9d^2, c^2+3cd.
4. $t^2-tr-20r^2$, t^2-25r^2, t^2-16r^2.
5. $4r^2-4r^4$, $1-2r+r^2$, r^2-1.
6. r^2-r-2, $r^2-8r+12$, r^2-5r-6.
7. r^3-8, $r^3-6r^2+12r-8$, $5r^2+10r+20$.
8. k^2-k-6, k^2-6k+9, $6k-18$.
9. n^5-32, $4-n^2$, $5n^2+10n$, $5n-10$.
10. m^3-8, $4-m^2$, $m^3+4mr^2+2m^2r$.
11. a^3-7a+6, a^2-3a+2, a^2+2a-3.

44. Equivalent fractions. Two fractions are **equivalent** if one can be obtained from the other either by multiplying

or by dividing both numerator and denominator by the same expression.

Two fractions having unlike denominators cannot be added or subtracted until they have been reduced to respectively equivalent fractions having like denominators.

EXAMPLE

Reduce to respectively equivalent fractions having the lowest common denominator:

$$\frac{2}{15\,a'}\quad\frac{5}{25\,a^2-4},\text{ and }\frac{10\,a}{25\,a^2-10\,a}.$$

Solution. Rewriting with denominators in factored form, we have

$$\frac{2}{15\,a'}\quad\frac{5}{(5\,a+2)(5\,a-2)},\text{ and }\frac{10\,a}{5\,a(5\,a-2)}\text{ or }\frac{2}{5\,a-2}.$$

The L. C. D. is $15\,a(5\,a-2)(5\,a+2)$.

Then
$$\frac{2}{15\,a}=\frac{2(5\,a+2)(5\,a-2)}{15\,a(5\,a-2)(5\,a+2)}.$$

$$\frac{5}{(5\,a+2)(5\,a-2)}=\frac{5\cdot 15\,a}{15\,a(5\,a-2)(5\,a+2)},$$

and
$$\frac{2}{(5\,a-2)}=\frac{2\cdot 15\,a(5\,a+2)}{15\,a(5\,a-2)(5\,a+2)}.$$

To change two or more fractions (in their lowest terms) to respectively equivalent fractions having the lowest common denominator we have the

RULE. *Rewrite the fractions with their denominators in factored form.*

Find the L. C. M. of the denominators of the fractions.

Multiply the numerator and the denominator of each fraction by those factors of this L. C. M. which are not found in the denominator of the fraction.

EXERCISES

Change to respectively equivalent fractions having the lowest common denominator:

1. $\frac{2}{5x}, \frac{3}{2x^2}$.

2. $\frac{3}{8xy}, \frac{x}{5x^2}$.

3. $\frac{x-2}{6x^3y^4}, \frac{2}{3x^5y}$.

4. $\frac{2x}{x-2}, \frac{3}{x+2}$.

5. $\frac{3}{x^2-9}, \frac{2x}{x-3}$.

6. $\frac{3a}{(a-2)^2}, \frac{2a^2}{a-2}$.

7. $\frac{2a^2}{2a+4}, \frac{a+3}{3a+1}, \frac{2a^2+5}{6a^2+14a+4}$.

8. $\frac{3x+y}{x-y}, \frac{1}{4(x-y)^2}, xy$.

9. $\frac{2x+1}{x^2-2x+4}, \frac{5}{x^3+8}, \frac{1}{x+2}$.

10. $\frac{2x-9}{x^5-32}, \frac{-2}{-x^3+2x^2}, \frac{x-2}{x^5+2x^4+4x^3+8x^2+16x}$.

11. $\frac{ax-bx-ar+br}{ax+bx+ar+br}, \frac{x^2-r^2}{a^2-b^2}, \frac{a^2-b^2}{x^2-r^2}$.

12. $\frac{2a}{8-a^3}, \frac{5a}{a-2}, \frac{4}{a^2-4}, \frac{3a+2}{a^3+2a^2+4a}$.

45. Addition and subtraction of fractions. To find the algebraic sum of two or more fractions in their lowest terms we proceed as in the

EXAMPLE

Find the algebraic sum of $\frac{3}{x^2}+\frac{a}{2x}-\frac{4-3a^2}{2ax}$.

Solution. The lowest common denominator is $2ax^2$.

Rewriting with common denominators, and adding numerators, we have

$$\frac{6a}{2ax^2}+\frac{a^2x}{2ax^2}-\frac{4x-3a^2x}{2ax^2}=\frac{6a+a^2x-(4x-3a^2x)}{2ax^2}$$

$$=\frac{6a+a^2x-4x+3a^2x}{2ax^2}$$

$$=\frac{6a+4a^2x-4x}{2ax^2}=\frac{3a+2a^2x-2x}{ax^2}.$$

Check. Substituting 2 for a and 3 for x, we have

$$\frac{3}{9}+\frac{2}{6}-\frac{-8}{12}=\frac{6+24-6}{18}.$$

$$\tfrac{4}{3}=\tfrac{4}{3}.$$

For finding the algebraic sum of two or more fractions we have the

RULE. *Reduce the fractions to respectively equivalent fractions having the lowest common denominator. Write in succession over the lowest common denominator the numerators of the equivalent fractions, inclosing each polynomial numerator in a parenthesis preceded by the sign of the corresponding fraction.*

Rewrite the fraction just obtained, removing the parentheses in the numerator.

Then combine like terms in the numerator and, if necessary, reduce the resulting fraction to its lowest terms.

CHECK. *Set the original expression equal to the final result and substitute in each member numeric values for the letters involved. The equation should be an identity.*

EXERCISES

Find the algebraic sum of:

1. $\frac{1}{2}+\frac{3}{4}+\frac{5}{8}+\frac{1}{16}$.

2. $\frac{2a}{5}+\frac{3a}{10}-\frac{2a}{15}$.

3. $\frac{3}{r} + \frac{5}{2r} - \frac{11}{3r}$.

4. $\frac{5}{4x} - \frac{6}{x} + \frac{3}{5x}$.

5. $\frac{2c}{3a} - \frac{5c^2}{6a^2} + \frac{c}{9a}$.

6. $\frac{4c}{3a^2} - \frac{5c^2}{2a^2} + \frac{6c}{5a^3}$.

7. $\frac{x^2}{3} + \frac{x^3}{2} - \frac{5x^3}{4}$.

8. $\frac{2}{a^2} + \frac{3}{a} - \frac{5}{2a^3} + \frac{1}{2a^2}$.

9. $\frac{5a}{a-3} - \frac{2}{3}$.

10. $\frac{7b}{2x} - \frac{b}{10x} + \frac{3b-1}{5^2x^2}$.

11. $\frac{6}{x} + \frac{3}{x-y} - \frac{6b}{x(x-y)}$.

12. $\frac{2k-3}{k(k-2)} - \frac{k^2-7}{8-k^3}$.

13. $\frac{s}{s-1} - \frac{2}{3} - \frac{5s-2}{s^2-1}$.

14. $2 + \frac{5r}{r-7}$.

HINT. $2 + \frac{5r}{r-7} = \frac{2}{1} + \frac{5r}{r-7}$.

15. $x + 3 - \frac{x^2}{x-3}$.

16. $a^2 - a + 1 - \frac{1}{a+1}$.

17. $a - \frac{a-6}{2a-3} - 2$.

HINT. $a - \frac{a-6}{2a-3} - 2 = a - 2 - \frac{a-6}{2a-3}$.

18. $\frac{7}{x^2-49} - \frac{3}{x^2-6x-7}$.

19. $\frac{4r-3}{r} + \frac{r-1}{2r-1} - \frac{5r^2+1}{2r^2-r}$.

20. $\frac{r-y}{3r+2y} + \frac{12ry}{9r^2-4y^2} - \frac{r+y}{3r-2y}$.

21. $\frac{-4cx}{x^3-c^3} - \frac{2x}{x^2-c^2} - \frac{1}{x-c}$.

22. $\frac{k-3}{k^2-2k-3} - \frac{2-k}{k^2-3k+2} - \frac{1}{1-k^2}$.

23. $\dfrac{b^2 - 16\,b}{4 - b^2} - \dfrac{3 + 2\,b}{b - 2} + \dfrac{3\,b - 2}{b + 2}$.

24. $\dfrac{5}{1 + k} + \dfrac{k}{1 - k^2} + \dfrac{2\,k}{k^2 - 1}$.

25. $\dfrac{s}{1 + s} - \dfrac{1 - s}{s} + \dfrac{2\,s}{s^2 - 1} - \dfrac{1}{1 - s}$.

26. $\dfrac{2}{r^2 + 1} - \dfrac{3}{r^2 - 1} - \dfrac{5}{1 - r^2} + \dfrac{4}{1 - r}$.

27. $\dfrac{a + 2}{a^2 - 7\,a + 12} - \dfrac{a + 2}{6 + a^2 - 5\,a} - \dfrac{1}{6\,a - a^2 - 8}$.

28. $\dfrac{a + b}{(b - c)(c - a)} + \dfrac{b + c}{ac - a^2 - bc + ab} + \dfrac{a + c}{(a - b)(b - c)}$.

29. $\dfrac{x^2 - (a - b)^2}{(x + b)^2 - a^2} + \dfrac{a^2 - (x - b)^2}{(x + a)^2 - b^2} + \dfrac{b^2 - (x - a)^2}{(a + b)^2 - x^2}$.

46. Multiplication of fractions. In algebra, as in arithmetic, the product of two or more fractions is the product of their numerators divided by the product of their denominators.

Thus $\frac{2}{3} \cdot \frac{5}{9} = \frac{10}{27}$.

Similarly, $\dfrac{a}{b} \cdot \dfrac{c}{d} = \dfrac{ac}{bd}$,

and $4 \cdot 2\frac{1}{8} = \dfrac{4}{1} \cdot \dfrac{17}{8} = \dfrac{17}{2}$.

Likewise $n \cdot \dfrac{a}{b} = \dfrac{n}{1} \cdot \dfrac{a}{b} = \dfrac{na}{b}$.

Integral and mixed expressions are reduced to fractional form before the multiplication is performed. Factors common to any numerator and any denominator are canceled the same number of times from each.

EXAMPLE

Multiply $3x^2\left(3 - \frac{1}{x}\right)\left(\frac{5m}{9x^2 - 1}\right)$.

Solution. Writing the above in fractional form, we have

$$\frac{3x^2}{1} \cdot \frac{3x - 1}{x} \cdot \frac{5m}{9x^2 - 1}.$$

Factoring and canceling,

$$\frac{\cancel{3}\,x}{\cancel{3x^2}} \cdot \frac{\cancel{(3x - 1)}}{\cancel{x}} \cdot \frac{5m}{(3x + 1)\cancel{(3x - 1)}} = \frac{15mx}{3x + 1}.$$

To find the product of two or more fractions or mixed expressions we have the

RULE. *If there are integral or mixed expressions, reduce them to fractional form.*

Separate each numerator and each denominator into its prime factors.

Cancel the factors (factor for factor) common to any numerator and any denominator.

Write the product of the factors remaining in the numerator over the product of the factors remaining in the denominator.

CHECK. *Set the given expression equal to the final result and substitute in each member numeric values for the letters. Simplify each member. The result should be an identity.*

47. **Division of fractions.** For division of fractions we have the

RULE. *Reduce all integral or mixed expressions to fractional form.*

Then invert the divisor or divisors and proceed as in multiplication of fractions.

Check as usual.

EXERCISES

Perform the indicated operations:

1. $\frac{3\,a}{25\,x^2m} \cdot \frac{10\,xm^2}{2\,a^2x}$.

2. $2\,c \cdot \frac{3\,b}{5\,cb^2} \cdot \frac{4\,c}{9\,b}$.

3. $\frac{3\,ab^2}{5\,cd^2} \cdot \frac{30\,c^3d}{6\,a^2b}$.

4. $\frac{8\,r^2s}{t^4} \cdot \frac{3\,s^2}{6\,xs^2} \cdot \frac{20\,r^2x^2}{5\,r^3t^3}$.

5. $4\frac{1}{2} \cdot \frac{2\,m^2}{3\,t^3} \cdot \frac{t^2}{15\,m} \cdot \frac{5\,tm}{mt^2}$.

6. $20\,a^2 \cdot \frac{14\,x}{4\,a^6} \cdot \frac{a^2}{42\,x^4}$.

7. $\frac{5\,k^3m}{9\,m^3s^2} \cdot \frac{6\,ms}{5\,k^2s} \cdot \frac{3\,k^2s^3}{3\,km^4}$.

8. $\frac{n^2-9}{2\,n^2} \cdot \frac{4\,n}{2\,n-6}$.

9. $\frac{2\,x^2+6}{5\,z^3} \cdot \frac{10\,z^4}{3\,x^2+9}$.

10. $\frac{a^2-1}{2\,a} \cdot \frac{6\,a^3}{a^2-2\,a+1} \cdot \frac{5\,a-5}{3(1-a)}$.

11. $\frac{r^2-4}{y-z} \cdot \frac{3\,y-3\,z}{r^2+6\,r+8}$.

12. $\frac{r^4-16}{(r^2-4)^2} \cdot \frac{r^2-4}{r^2+4}$.

13. $\left(3\,a+\frac{1}{a}\right)\left(\frac{a^2}{9\,a^4-1}\right)\left(a-\frac{1}{3\,a}\right)$.

14. $\frac{2\,r^2-13\,r+15}{4\,r^2-9} \cdot \frac{2\,r+1}{2\,r-1} \cdot \frac{2\,r-1}{r-5}$.

15. $\frac{s^2-11\,s+30}{s^3-6\,s^2+9\,s} \cdot \frac{s^2-3\,s}{s^2-25} \cdot \frac{s^2+2\,s-15}{s^2-9\,s}$.

16. $\left(4+\frac{3}{r^2-1}\right)\left(\frac{3\,r}{2\,r-1}-1\right)$.

17. $\frac{120\,r^2s}{42\,r^3b} \div \frac{100\,rs^3}{147\,r^2x^2}$.

18. $\frac{24\,r^2}{99\,ry^2} \cdot \frac{8\,y^2z}{22\,r^2y} \div \frac{48\,r^3z}{121\,r^2y^3}$.

NOTE. See Rule Section 2.

19. $\frac{33\,m^2n^2}{5\,x^3} \div \frac{52\,m^5n}{5\,x^2} \div \frac{297\,n^2x}{39\,m^3x}$.

20. $\frac{5\,x^2}{x-y} \cdot \frac{(x-y)^2}{20\,x^2} \div \frac{1}{x(x-y)}$.

21. $\frac{r^2 - rx - 6\,x^2}{r^3 - 9\,rx^2} \div \frac{r+2\,x}{r+3\,x}$.

22. $\frac{x^2 - 8\,x + 15}{6 - 5\,x + x^2} \div \frac{7\,x - 10 - x^2}{4 - 4\,x + x^2}$.

23. $\frac{r^4 + 64\,s^4}{r^2 - s^2} \cdot \frac{r+s}{r^2 - 4\,rs + 8\,s^2} \div \frac{r^2 + 4\,rs + 8\,s^2}{r^2 - 2\,rs + s^2}$.

24. $\frac{r^5 - 1}{r+2} \div \frac{r^3 - 1}{r^2 + 5\,r + 6} \cdot \frac{r^3 + r^2 + r}{r^4 + r^3 + r^2 + r + 1}$.

Simplify each term in Exercises 25–26:

25. $\left(\frac{3\,r^2}{2\,s}\right)^4 - 4\left(\frac{3\,r^2}{2\,s}\right)^3\left(\frac{2\,s^2}{3\,r}\right) + 6\left(\frac{3\,r^2}{2\,s}\right)^2\left(\frac{2\,s^2}{3\,r}\right)^2$.

26. $\left(\frac{2\,a}{5\,x^2}\right)^3 - 3\left(\frac{2\,a}{5\,x^2}\right)^2\left(\frac{5\,x^3}{4\,a^2}\right) + 3\left(\frac{2\,a}{5\,x^2}\right)\left(\frac{5\,x^3}{4\,a^2}\right)^2$.

Perform the indicated operations:

27. $\left(\frac{s}{3} - \frac{3}{s}\right) \div \left(\frac{s^2 - 6\,s + 9}{3\,s}\right)$.

28. $\left(3\,r - \frac{2\,rs + s^2}{r}\right) \div \left(\frac{r^2 - rs}{r^2s^2}\right)$.

29. $\frac{a}{a^2 - 4} \div \left(\frac{a}{3\,a^2 + 7\,a + 2}\right)\left(3\,a - \frac{2}{a} - 5\right)$.

30. $\dfrac{x+1+\dfrac{4}{x+3}}{x-5+\dfrac{12}{x+3}}$.

HINT. An expression of this form is called a complex fraction. It is simply another way of writing

$$\left(x+1+\frac{4}{x+3}\right)\div\left(x-5+\frac{12}{x+3}\right).$$

31. $\dfrac{2+\dfrac{x+6}{x-2}}{3+\dfrac{5}{x-2}}$.

32. $\dfrac{x-1-\dfrac{3}{x}}{x+1+\dfrac{x}{x-3}}$.

33. $\dfrac{\dfrac{a}{a+1}+\dfrac{1-a}{a}}{\dfrac{a}{a+1}-\dfrac{1-a}{a}}$.

34. $\dfrac{\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}}{1+\dfrac{2a^2b^2+2b^4}{a^4-b^4}}$.

35. $\dfrac{\left(a-\dfrac{1}{a^2}\right)^2\left(\dfrac{2}{a-1}\right)}{\left(a+1+\dfrac{1}{a}\right)}$.

36. $\dfrac{\dfrac{2}{b}-\dfrac{1}{a+b}+\dfrac{1}{a-b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

37. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{\dfrac{b}{a+b}}{1+\dfrac{2b-a}{a-b}}$.

38. $\dfrac{2}{\dfrac{1}{b^2}\left(\dfrac{1}{c}-\dfrac{b}{2a}\right)}-b^2c\left(1-\dfrac{bc}{a-\dfrac{bc}{2}}\right)$.

39. If $x = \dfrac{a}{b+c}$, $y = \dfrac{b}{a+c}$, and $z = \dfrac{c}{a+b}$, find the value of $\dfrac{x}{x+1} + \dfrac{y}{y+1} + \dfrac{z}{z+1}$.

40. What value has $\dfrac{m^3 - n^3}{m^2 + n^2}$ when $m = a - \dfrac{1}{a}$ and $n = a + \dfrac{1}{a}$?

41. $1 - \dfrac{1}{1 - \dfrac{1}{1 + \frac{1}{2}}}$.

HINT. First simplify $1 + \frac{1}{2}$. Then $1 - \frac{1}{\frac{3}{2}}$, etc.

42. $1 - \dfrac{1}{1 + \dfrac{2}{1 - \frac{2}{3}}}$.

43. $1 + \dfrac{1}{1 - \dfrac{1}{1 + \frac{1}{a}}}$.

44. $\dfrac{\left(1 + \dfrac{x}{a+b} + \dfrac{x^2}{(a+b)^2}\right) \cdot \left(1 - \dfrac{x^2}{(a+b)^2}\right)}{\left(1 - \dfrac{x^3}{(a+b)^3}\right)\left(1 + \dfrac{x}{a+b}\right)}$.

48. Equations involving fractions. Equations involving fractions are solved as in the following

EXAMPLE

Solve: $\dfrac{4}{n} - \dfrac{3(2-n)}{n-1} = 3.$ (1)

Solution. The L. C. M. of the denominators is $n(n-1)$.

$(1) \cdot n(n-1)$, $4(n-1) - 3n(2-n) = 3n(n-1)$. (2)

From (2), $4n - 4 - 6n + \cancel{3n^2} = \cancel{3n^2} - 3n.$

$n = 4.$

Check. Substituting 4 for n in (1),

$1 + 2 = 3.$

For solving equations in one unknown which may or may not involve simple fractions we have the

RULE. *Where polynomial denominators occur, factor them if possible.*

Find the L. C. M. of the denominators of the fractions and multiply each fraction and each integral term of the equation by it, using cancellation whenever possible.

Transpose and solve as usual.

Reject all values for the unknown which do not satisfy the original equation.

NOTE ON INVERTING THE DIVISOR. The student of algebra will remember that in arithmetic the divisor is inverted in division of fractions, but he may fail to recall just why. It is important to understand thoroughly this particular step of the process of division of fractions and why division is then replaced by multiplication. In a book on arithmetic the explanation could be made clear by analyzing the solution of a simple problem like $8 \div \frac{3}{5}$. To see just why the divisor is here inverted consider first $1 \div \frac{3}{5}$. This last requires that a unit of one kind (unity itself) be divided by three units of a different kind (fifths). Any difficulty of this nature is avoided by reducing the number 1 to fifths; then $1 \div \frac{3}{5}$ becomes $\frac{5}{5} \div \frac{3}{5}$, and the problem is transformed to that of dividing three units of a certain kind (fifths) into five of the same kind. This can be done directly and we have $\frac{5}{5} \div \frac{3}{5}$. This is the same as $5 \div 3$ or $\frac{5}{3}$. Hence the correct answer $\frac{5}{3}$ is $\frac{3}{5}$ inverted. This means that to find how often $\frac{3}{5}$ is contained in the number 1 we merely invert it, multiply by 1 and obtain $\frac{5}{3}$. Obviously $\frac{3}{5}$ is contained in 2 twice as often as in 1, in 3 three times as often, and in 8 eight times as often, etc. Hence $8 \div \frac{3}{5} = 8 \times \frac{5}{3} = \frac{40}{3}$.

In the more general problem $1 \div \frac{a}{c}$ we may as before write

$$1 \div \frac{a}{c} = \frac{c}{c} \div \frac{a}{c} = \frac{c}{a}.$$

That is, $\frac{a}{c}$ is contained in the number 1, $\frac{c}{a}$ times. In the general problem $n \div \frac{a}{c}$, it will be contained n times as often as in 1.

Hence $$n \div \frac{a}{c} = n \times \frac{c}{a} = \frac{nc}{a}.$$

EXERCISES

Solve and check:

1. $\frac{7x}{3} - 1 = \frac{5x}{3} + 3.$

2. $\frac{2x}{5} + 1 = x - \frac{7}{5}.$

3. $\frac{11x}{9} - \frac{2}{3} = x.$

4. $\frac{3r}{2} - \frac{2r}{3} = \frac{5}{2}.$

5. $\frac{3r}{2} - \frac{16}{3} = r - \frac{25}{6}.$

6. $\frac{5x}{6} - \frac{21}{5} = \frac{5x - 2}{10}.$

7. $\frac{3r + 14}{3} - \frac{6 - r}{4} = -1.$

8. $\frac{2(k - 4)}{3} - 1 = \frac{k + 5}{3}.$

9. $\frac{3}{4x} + \frac{7}{16} = \frac{4}{3x}.$

10. $\frac{8}{7x + 3} = \frac{3}{3x + 1}.$

11. $\frac{2x - 7}{3x - 4} + \frac{3}{2} = 0.$

12. $\frac{3x - 3}{x + 7} + \frac{3}{5} = 0.$

13. $(x + 5)(x + 1) - (x - 3)(x - 2) = 10.$

14. $\frac{3(1 - 8x)}{5x} - \frac{2(1 + 8x)}{8x - 1} = \frac{1 - 34x}{5x}.$

15. $\frac{3}{2x + 1} + \frac{2x + 1}{1 - 2x} = 1 - \frac{8x^2}{4x^2 - 1}.$

16. $\frac{3r - 5}{5r - 5} + \frac{5r - 1}{7r - 7} - \frac{r - 4}{1 - r} = 2.$

17. $\frac{x + a}{a} - \frac{x + b}{b} = b - a.$

18. $\frac{x}{a} - a = \frac{x}{b} - b.$

19. $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}.$

20. $\frac{x}{a - 1} + \frac{x}{a + 1} = 2a.$

21. $\frac{x}{a + 2} - \frac{x + 16}{a - 2} = 4a.$

22. $\frac{2x+1}{x+3} + \frac{3x-7}{2-x} = \frac{9-3x-x^2}{x^2+x-6}$.

23. $\frac{x+cd}{c+d} - \frac{x-cd}{c-d} = \frac{2c^2d-2cd^2}{c^2-d^2}$.

24. $\frac{x-a}{2x-a} - \frac{3x-c}{6x-c} = 0$.

25. $\frac{2r}{2-.4r} - \frac{5}{2} = \frac{2r}{4-.8r} - \frac{2r}{1-.2r}$.

26. $\frac{3s+4a}{s+2a} + \frac{3s-5a}{4a-s} = \frac{10a^2}{s^2-2as-8a^2}$.

27. $\frac{5.3r-3.7}{13} = \frac{12.5r-8.06}{14}$.

28. $\frac{5x+.4}{.6} + \frac{1.3x-.05}{4} = \frac{30.35-8x}{2.4}$.

29. $12.2 - \frac{5-2x}{.5} = 2.3 - (5+7x) + \frac{2-4x}{2}$.

30. $\frac{3x-1}{.25} + \frac{x-4}{.5} = 3(3x-14)$.

31. $\frac{2r-3}{.3r-.4} = \frac{.4r-.6}{.06r-.07}$.

32. $\frac{x-\frac{1}{3}}{x+\frac{2}{3}} = \frac{x+1}{x+\frac{4}{3}}$.

33. $\frac{m+\frac{1}{2}}{m-\frac{1}{4}} = \frac{m-\frac{1}{2}}{m+\frac{1}{4}}$.

PROBLEMS

1. Separate 175 into two parts such that one part will be $\frac{2}{5}$ of the other.

2. Separate 360 into two parts such that the greater will be $3\frac{1}{11}$ times the less.

3. Separate 336 into two parts such that $\frac{1}{3}$ of the greater shall equal $\frac{3}{5}$ of the less.

4. By what number must 875 be divided so as to give the partial quotient 32 and a remainder 11?

5. Separate 175 into two parts such that their quotient is $5\frac{1}{4}$.

6. One third of a pile is in earth, one fifth is in water, and 14 feet are above water. How long is the pile?

7. A certain pile is 57 feet long. The part in earth is $\frac{2}{3}$ the part in water; the part in water is $\frac{2}{3}$ the part in air. How many feet of it are in water? in earth? in air?

8. A boy is 14 years old and his sister is 9 years old. In how many years will the boy be $\frac{7}{6}$ as old as his sister?

9. The square of a certain number is 785 greater than $\frac{8}{9}$ the product of the next two consecutive numbers. Find the number.

10. The denominator of a fraction is 4 times the numerator. If the numerator is increased by 4 and the denominator is decreased by 4, the resulting fraction is $\frac{3}{2}$. What are the terms of the fraction?

11. A man sold $\frac{1}{3}$ of his salt and $\frac{2}{3}$ pound more. He then sold $\frac{1}{5}$ of what remained and $\frac{4}{5}$ pound more; he then sold $\frac{1}{3}$ of the rest and had 8 pounds left. How much had he at first?

12. A rectangle is $3\frac{1}{3}$ times as long as it is wide. If it were 8 yards shorter and $2\frac{1}{2}$ yards wider, its area would be 108 square feet less. Find the dimensions of the rectangle.

13. A square court has $\frac{9}{8}$ the area of a rectangular court whose length is 5 yards greater and whose width is 2 yards less than the side of the square. What are the dimensions of the square court?

14. A, B, and C together have \$5460. A has \$50 more than $\frac{2}{3}$ as much as B, and C has \$45 more than $\frac{3}{4}$ as much as B. How much has each?

15. If a man can do in 3 days a piece of work which takes a boy 7 days, how long will it take both working together to complete the work?

16. A can do a piece of work in 8 days and B in 10 days. How many days will they both require working together?

17. A can do a piece of work in 10 days, and B in 15 days. After they have worked together 5 days, how long will it take A to finish it alone? B alone?

18. A can do a piece of work in 6 days, B in 8 days, and C in 9 days. A works 2 days and stops; B works 2 days, when C joins him. How long must C work before the job is completed?

19. A tank has a supply pipe that fills it in 5 hours and a waste pipe that empties it in 7 hours. If the tank is empty and both pipes are open, how much time must elapse before the tank is filled?

20. One pipe can fill a swimming tank in 27 minutes and another pipe can fill it in 30 minutes. A waste pipe can empty the tank in 20 minutes. With all pipes open, how long will it be before the tank is filled?

21. The sum of the numerator and denominator of a certain fraction is 36. If 2 be added to the numerator and subtracted from the denominator, the resulting fraction is $\frac{1}{3}$. What was the original fraction?

22. The width of a rectangle is $\frac{2}{3}$ its length. The perimeter is 150 feet. What are the dimensions of the rectangle?

23. The diameter of the earth is $3\frac{2}{3}$ times that of the moon, and the difference of the two diameters is 5760 miles. Find each diameter in miles.

24. The diameter of the sun is 3220 miles greater than 109 times the diameter of the earth, and the sum of the diameters is 874,420 miles. Find each diameter in miles.

25. A man rowed 12 miles up a river in 5 hours and back in 3 hours. What is his rate in still water?

26. A man who can row 4 miles per hour in still water finds that it requires $5\frac{1}{2}$ hours to row upstream a distance which it requires $2\frac{1}{2}$ hours to row down. Find the speed of the current.

27. A passenger train whose speed is 45 miles per hour leaves a station 3 hours and 45 minutes after a freight train. The passenger train overtakes the freight in 4 hours and 15 minutes. Find the speed of the freight train in miles per hour.

28. A, B, and C can do a piece of work in 12 days, B and C together in 20 days, and C alone in 45 days. How long does it take A alone to do the work? B alone?

29. A man invests a part of \$9000 at 5% and the remainder at $4\frac{1}{2}\%$. If the yearly interest on the whole investment is \$423, how much was invested at each rate?

30. A man invests \$6800 in two parts: the first part at $5\frac{1}{2}\%$, the second at $7\frac{1}{5}\%$. If the average rate of interest is $6\frac{9}{20}\%$, find the amount of each investment.

31. How much water must be added to a gallon of alcohol 90% pure so as to make a mixture 10% pure?

HINT. Let w = the number of gallons of water to be added.

Then $$\frac{\frac{90}{100} \cdot 1}{1 + w} = \frac{10}{100}.$$

32. How much water must be added to 25 gallons of milk containing 8% butter fat to make a mixture containing 5% butter fat?

33. It is desired to mix coffee which sells for 35¢ per pound with coffee which sells for 50¢ a pound so as to obtain a ten-pound mixture which may be sold for 45¢ a pound. How many pounds of each kind of coffee must be used?

34. How many quarts of 45¢ vinegar must be mixed with 10 quarts of 25¢ vinegar so that a quart of the mixture may sell for 40¢?

35. A certain lot of pig iron contains 91% iron. How much pure iron must be melted with 10 tons of pig iron to make iron 97% pure?

36. The arms of a lever are 4 feet and 5 feet in length respectively. Excluding the weight of the beam, what weight on the longer arm will balance 90 pounds on the shorter?

37. The arms of a balanced lever are 8 feet and 13 feet respectively. The shorter arm carries 39 pounds. Find the load on the longer arm.

38. If the load on the longer arm in Problem 37 be reduced 8 pounds, how many feet from the fulcrum must a 26-pound weight be placed on the shorter arm to restore the balance?

39. How many ounces of water must be added to 1 ounce of carbolic acid which is 90% pure to make a 3% solution?

40. At what time between 4 and 5 o'clock will the hands of a clock be together?

Solution. The minute hand moves twelve times as fast as the hour hand. While the minute hand travels x spaces, the hour hand travels $\frac{x}{12}$ spaces. Hence, $x - \frac{x}{12}$ equals the number of spaces gained by the minute hand in any given time x.

In the time from 4 o'clock until the hands are together, the minute hand must gain 20 minute spaces on the hour hand to overtake it.

Therefore $$x - \frac{x}{12} = 20.$$

Whence $$x = 21\tfrac{9}{11}.$$

Hence, the hands are together $21\frac{9}{11}$ minutes after 4 o'clock.

41. At what time between 9 and 10 o'clock are the hands of a clock together?

42. At what time between 4 and 5 o'clock are the hands of a clock in a straight line?

43. At what time between 7 and 8 o'clock will the minute hand be 10 minute spaces ahead of the hour hand? Ten minute spaces behind?

44. At what time between 5 and 6 o'clock are the hands of a clock at right angles?

CHAPTER V

LINEAR SYSTEMS

49. Graphical solution of a linear system in two unknowns. The construction of the graph of a single linear equation in two unknowns or of a linear system in two unknowns depends on several assumptions and definitions. These are:

I. To have at right angles to each other two lines, $X'OX$, called the ***x*-axis**, and $Y'OY$, called the ***y*-axis**.

II. To have a unit of distance on these lines.

Thus the number 2 will correspond to a distance of twice the unit, the number $4\frac{1}{2}$ to a distance $4\frac{1}{2}$ times the unit, etc.

III. That the distance (measured parallel to the x-axis) from the y-axis to any point in the paper be the ***x*-distance** (or abscissa) of the point, and the distance (measured parallel to the y-axis) from the x-axis to the point be the ***y*-distance** (or ordinate) of the point.

IV. That the x-distance of a point to the *right* of the y-axis be represented by a *positive number*, and the x-distance of a point to the *left* by a *negative* number; also that the y-distance of a point *above* the x-axis be represented by a *positive* number, and the y-distance of a point *below* the x-axis by a negative number. Briefly, *distances measured from an axis to the* **right** *or* **upward** *are* **positive**; *to the* **left** *or* **downward** *are* **negative.**

V. That every point in the surface of the paper corresponds to a *pair of numbers*, one or both of which may be positive, negative, integral, or fractional.

VI. That of a given pair of numbers the first be the measure of the x-distance and the second the measure of the y-distance.

Thus the point (2, 3) is the point whose x-distance is 2 and whose y-distance is 3.

The point of intersection of the axes is called the **origin.**

The values of the x-distance and the y-distance are often called the **coördinates** of the point.

The relation between an equation and its graph may be stated as follows:

The equation of a line is satisfied by the values of the x-distance and the y-distance of any point on that line.

Any point, the values of whose x-distance and whose y-distance satisfy the equation, is on the graph of the equation.

The graph of a linear equation in two unknowns is a straight line. Therefore it is necessary in constructing the graph of such an equation to locate only two points whose coördinates satisfy the equation and then to draw through the two points a straight line. It is usually most convenient to locate the two points where the line cuts the axes. If these two points are very close together, however, the direction of the line will not be accurately determined. This error can be avoided by selecting two points at a greater distance apart.

The *graphical solution* of a linear system in two unknowns consists in plotting the two equations to the same scale and on the same axes and obtaining from the graph the values of x and y at the point of intersection of the lines.

Through the graphical study of equations we unite the two subjects of geometry and algebra, which seem very different, and learn to interpret problems of the one in the language of the other.

Example

Solve graphically the system

$$3x - 4y = 24, \quad (1)$$

$$x + 3y = -5. \quad (2)$$

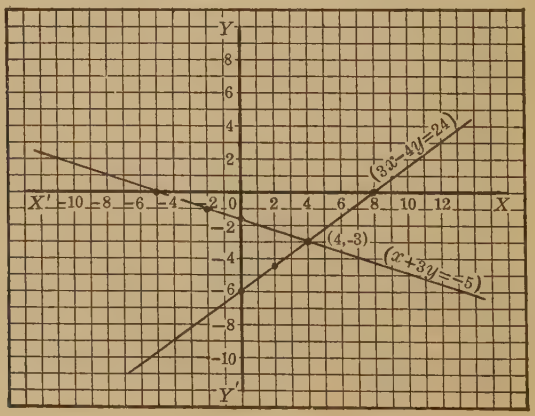
Solution. Substituting zero for x in $3x - 4y = 24$, we obtain $y = -6$. Substituting zero for y, we obtain $x = 8$, and 2 for x we obtain $y = -\frac{9}{2}$. This may be expressed in tabular form:

If $x =$	0	8	2
Then $y =$	-6	0	$-\frac{9}{2}$

Similarly, for the equation $x + 3y = -5$, we obtain the following table:

If $x =$	0	-5	-2
Then $y =$	$-\frac{5}{3}$	0	-1

Then, constructing the graph of each equation as indicated in the figure below, we obtain for the coördinates of the point of intersection of the two lines, $x = 4$, and $y = -3$.



Check. Substituting 4 for x and -3 for y in (1) and (2) respectively we have

$$12 - (-12) = 24$$

and

$$4 - 9 = -5.$$

$$-5 = -5$$

EXERCISES

Solve graphically and check:

1. $2x + y = 8,$
   $x + 2y = 13.$
2. $x - y = 6,$
   $3x + 4y = -17.$
3. $3x - 2y = 6,$
   $4x + 3y = 25.$
4. $x + 2y = 4,$
   $6y + 2x = 20.$
5. $4x - 6y = 0,$
   $5y - 3x = -1.$
6. $5x - y = 8,$
   $x - 5y = 16.$
7. $x + y = 5,$
   $y + 2 = 0.$
8. $x + 5 = -3y,$
   $6y + 2x - 11 = 0.$
9. $x + y = 4,$
   $x + 2y = 7.$
10. $x - y = 5,$
    $3x + 2y = 5.$

50. Elimination. In order to find values of x and y which satisfy the equation

$$2x + 5y = 15 \qquad (1)$$

when we know that

$$y = x - 4 \qquad (2)$$

we may substitute for y in the first equation the value of y from the second, obtaining the single equation in x,

$$2x + 5(x - 4) = 15, \text{ or } 7x = 35, \text{ or } x = 5.$$

The process by which we have obtained one equation containing one unknown from the two equations (1) and (2), each of which contains two unknowns, illustrates one method of elimination.

In general, the process of deriving from a system of n equations a system of $n - 1$ equations, containing one

variable less than the original system, is called **elimination**, since one variable is eliminated.

For example, if we have a system of two equations in two unknowns, the process of elimination leads to one equation in one unknown. Since we can always solve such an equation, it appears that we can solve a system of two equations in two unknowns whenever it is possible to eliminate one of the unknowns. We shall see that only in certain exceptional cases is elimination impossible. This is either because more than one unknown is removed when we try it or because the result of attempted elimination is not an equation.

Only two methods of solution will be considered — that involving **elimination by substitution** and that involving **elimination by addition or subtraction.**

51. Solution by substitution. The method of solving a system of two linear equations by substitution is illustrated in the

EXAMPLE

Solve the system $x - 2y = -8,$ (1)

$6x - y = 7.$ (2)

Solution. From (1) $x = 2y - 8.$ (3)

Substituting this value for x in (2),

$$6(2y - 8) - y = 7.$$

Simplifying, $12y - 48 - y = 7.$

Combining terms, $11y = 55.$

Whence $y = 5.$

Substituting 5 for y in (3), and solving,

$$x = 2.$$

Check. Substituting 2 for x and 5 for y in (1) and (2),

$$2 - 2(5) = -8, \text{ or } -8 = -8,$$

$$6 \cdot 2 - (5) = 7, \text{ or } 7 = 7.$$

The method of the preceding solution for solving a system of two linear equations is stated in the following

RULE. *Solve either equation for the value of one unknown in terms of the other.*

Substitute this value for the unknown which it represents, in the equation from which it was not obtained, and solve the resulting equation. Substitute the definite value just found in the simplest of the preceding equations which contains both unknowns, and solve, thus obtaining a definite value for the other unknown.

CHECK. *Substitute for each unknown in both original equations its value as found. If the resulting equations are not obvious identities, simplify them until they become so.*

EXERCISES

Solve by substitution:

1. $x + y = -1,$
   $5x - y = 13.$

2. $2x - 3y = 2,$
   $5y - 4x = -6.$

3. $4x - 5y = 6,$
   $3x + 7y = -17.$

4. $2x + y = 8,$
   $x - 2y = 9.$

5. $2(x + y) + 3y = 4,$
   $5 = x + y.$

6. $4x + 7 = 3y,$
   $x + y = 0.$

7. $6x + 38 = 12y,$
   $4x - 8y = 0.$

8. $4x - y = -2,$
   $5x + y = 6\frac{1}{2}.$

9. $3x + 2y = 1,$
   $2x - 3y = 5.$

10. $5x + \frac{3}{2}y = 7\frac{5}{6},$
    $x - 4y = -11\frac{1}{3}.$

52. Solution by addition or subtraction. The method of solving a system of two linear equations by addition or subtraction is illustrated in the following example.

EXAMPLE

Solve the system
$$5x - 2y = 25, \quad (1)$$
$$11x + 3y = 18. \quad (2)$$

Solution. First eliminate y, as follows:

$(1) \cdot 3$, $\quad 15x - 6y = 75, \quad (3)$

$(2) \cdot 2$, $\quad 22x + 6y = 36, \quad (4)$

$(3) + (4)$, $\quad 37x = 111. \quad (5)$

$(5) \div 37$, $\quad x = 3. \quad (6)$

Substituting 3 for x in (2), we obtain $y = -5$.

Check. Substituting 3 for x and -5 for y in (1) and (2),

$$15 + 10 = 25, \text{ or } 25 = 25,$$
$$33 - 15 = 18, \text{ or } 18 = 18.$$

Before starting to eliminate an unknown, the system should always be inspected to determine which unknown can be removed most conveniently. In this case the elimination of y is simpler, because it involves multiplication by smaller numbers than does the elimination of x.

The method of the preceding solution is stated in the following

RULE. *If necessary, multiply each member of the first equation by a number, and each member of the second equation by another number, such that the coefficients of the same unknown in the resulting equations will be numerically equal.*

If these coefficients have like signs, subtract one equation from the other; if they have unlike signs, add. Then solve the equation thus obtained. Substitute the value just found in the simplest of the preceding equations which contains both unknowns and solve for the other unknown.

CHECK. *As in section 51.*

ORAL EXERCISES

Solve by addition or subtraction:

1. $x + y = 2$, $x - y = 4$.
2. $2x + y = 4$, $3x - y = 1$.
3. $3x - y = 2$, $x + y = 10$.
4. $x + y = 7$, $y - x = 9$.
5. $x + 7y = -1$, $4x - 7y = 31$.
6. $x + y = 5$, $x - y = 1$.
7. $x + y = 6$, $x - y = 4$.
8. $2x + y = 13$, $2x - y = 7$.

53. Special cases. A system of equations which has a common set of roots is called a **simultaneous** system. The equation $x + y = 8$ has as roots any set of two numbers whose sum is 8. If $x + y = 4$ is taken as the other equation of a system, one can see immediately that the two equations have no set of roots in common, since the sum of two numbers cannot be 8 and 4 at the same time.

Graphs of these two equations give parallel lines. Since two parallel lines cannot intersect, we see clearly why no set of roots exists for $x + y = 8$ and $x + y = 4$.

A system of equations which does not have a common set of roots is called an **inconsistent** or **incompatible** system.

The attempt to solve an incompatible system results in getting rid not only of one but of both unknowns and leads to a statement in the form of an equation which is false.

Consider $x + y = 12$, (1)

$x + y = 6$, (2)

(1) − (2), $0 = 6$, which is false.

If, on the other hand, the equation $x + y = 12$ is taken for one equation of a system and $2x + 2y = 24$ for the other, it appears that any set of numbers which satisfies one equation satisfies also the other, since if the sum of two numbers is 12, the sum of twice those numbers is 24, and any one of the countless sets of roots of one equation is a set of roots of the other. In fact, the second equation may be obtained from the first by multiplying each member by 2.

If one equation of a system can be obtained from one or more of the other equations of the system by application of one or more of the axioms, it is called a **derived** or **dependent** equation. If it cannot be so obtained, it is called **independent.**

Thus equations (1) and (2) in the example on page 88 are independent, while equation (5) is derived from them.

The graphs of $x + y = 12$ and $2x + 2y = 24$ are identical lines. Hence any point on the first line is a point on the second.

EXERCISES

Solve the following systems by addition and subtraction:

1. $2x - y = 9,$
   $x + 7y = 12.$

2. $3x + 4y = -6,$
   $5x - 3y = 19.$

3. $5x + 7y = 1,$
   $8x + 9y = 6.$

4. $4x + 5y = -25,$
   $7x - 6y = -29.$

5. $3x - 4y = -4,$
   $x + y = 2\frac{1}{6}.$

6. $3x + 5y = 5,$
   $9x - 6y = -27.$

7. $5x + 7y = -32,$
   $x - 2y = 10\frac{3}{5}.$

8. $2x + 4y = -22,$
   $5x + y = 35.$

9. $7x + 10y = -50,$
$5y - 3x = 27.$

10. $4x + 3y = 4.1,$
$5x - 7y = -2.4.$

11. $\frac{r}{5} = \frac{s}{4},$
$3r + \frac{1}{2}s = 17.$

12. $\frac{3y+1}{4} - \frac{z+22}{12} = 3,$
$z - 2y = 1.$

13. $.5x + .7y = 9\frac{3}{10},$
$.8x - .2y = 3.$

14. $\frac{5m}{6} + \frac{n}{4} = 7,$
$\frac{2m}{3} - \frac{n}{8} = 3.$

15. $\frac{r+1}{3} = \frac{2-s}{2} - \frac{5}{6},$
$\frac{3r-2}{3} = \frac{s-3}{4} - \frac{5}{12}.$

16. $\frac{1}{x} + \frac{1}{y} = -\frac{1}{6},$
$\frac{2}{x} - \frac{3}{y} = \frac{4}{3}.$

HINT. Solve first for $\frac{1}{x}$ and $\frac{1}{y}$.

17. $\frac{2r+4s}{2r-s} = \frac{38}{3},$
$\frac{5}{r} - \frac{7}{s} = 0.$

18. $\dfrac{3}{x-3} - \dfrac{1}{y} = 0,$

$\dfrac{x-6y}{5} = \dfrac{2}{5}.$

19. $\dfrac{2.5h + 1 - .1k}{k - 10 + h} = 2.5,$

$\dfrac{1}{.8h - 2.2} + \dfrac{20}{35 - 5.5k} = 0.$

20. $\dfrac{3x + z}{z - x} + 2 = \dfrac{12}{x - z},$

$\dfrac{x+7}{z} + \dfrac{17 + x}{5 - z} = 0.$

Solve for x and y:

21. $2x - 3y = 9b,$
$5x + 9y = 6b.$

22. $3x + 2y = \dfrac{a}{5} + 5b,$

$2x + 3y = 5b - \dfrac{a}{5}.$

23. $5x + 4y = 10a + 4,$
$x - 2ay = 0.$

24. $7x + 5y = 21c,$
$\dfrac{x}{5} + \dfrac{y}{7} = 3c.$

25. $2ax - 3by = 7,$
$5ax + 7by = 3.$

26. $\dfrac{a}{x} + \dfrac{3a}{y} = 1,$

$\dfrac{3a}{x} + \dfrac{a}{y} = \dfrac{1}{2}.$

27. $\dfrac{x+y}{4a} = 3 + \dfrac{x-y}{7},$

$\dfrac{2x - y}{a} = 6.$

28. $\dfrac{x}{a} + \dfrac{y}{b} = \dfrac{a+b}{ab},$

$x - y = \dfrac{a^2 - b^2}{ab}.$

29. $2ax - 4by = 6c,$
$3ax - 5by = 5c.$

30. $ax + by = c,$
$kax + kby = ck.$

31. $ax + by = c,$
$dx + fy = g.$

54. Equations in several unknowns. We have already seen that the equation $x + y = 8$ is satisfied by an unlimited number of sets of roots, since there is an infinity of pairs of numbers whose sum is 8.

An equation or a system of equations which is satisfied by an infinite number of sets of roots is said to be **indeterminate.**

If a simultaneous system is satisfied by only a limited number of sets of roots, it is said to be **determinate.**

The system $x + y = 8$, $x - y = 4$ is determinate and has the set of roots (6, 2). The system $2x + 2y = 16$, $x + y = 8$, is indeterminate.

When the number of unknowns in a system of linear equations exceeds the number of equations, the system is indeterminate. If the number of equations equals the number of unknowns, the system is usually determinate and simultaneous. If the number of equations exceeds the number of unknowns, the system is usually inconsistent. There are many special cases which arise in the study of linear systems in n unknowns in section 55, but they become very complicated for larger values of n, and a thorough study of them is quite beyond the scope of this text.

NOTE. It is not a little remarkable that the writings of the first great algebraist, Diophantos of Alexandria (about A.D. 275), are devoted almost entirely to the solution of indeterminate equations; that is, to finding the sets of related values which satisfy an equation in two unknowns, or perhaps, two equations in three unknowns. We know practically nothing of Diophantos himself, except the information contained in his epitaph, which reads as follows: "Diophantos passed one sixth of his life in childhood, one twelfth in youth, one seventh more as a bachelor; five years after his marriage a son was born who died four years before his father, at half his father's age." From this statement the reader was supposed to be able to find at

what age Diophantos died. As a mathematician Diophantos stood alone, without any prominent forerunner or disciple, so far as we know. His solutions of the indeterminate equations were exceedingly skillful, but his methods were so obscure that his work had comparatively little influence upon later mathematicians.

55. Determinate systems. The method of obtaining the set of roots of a determinate system in three unknowns is illustrated in the

EXAMPLE

Solve the system $\begin{cases} x + 4y - 3z = 21, & (1) \\ 2x - 5y + z = -6, & (2) \\ 5x - 3y + 2z = -3. & (3) \end{cases}$

Solution. First eliminate one unknown, say z, between (1) and (2):

$$x + 4y - 3z = 21. \quad (1)$$

(2) · 3, $\quad 6x - 15y + 3z = -18. \quad (4)$

(1) + (4), $\quad 7x - 11y = 3. \quad (5)$

Now eliminate z between (2) and (3):

(2) · 2, $\quad 4x - 10y + 2z = -12. \quad (6)$

$$5x - 3y + 2z = -3. \quad (3)$$

(6) − (3), $\quad -x - 7y = -9. \quad (7)$

The equations (5) and (7) contain the same two unknowns x and y.

$$7x - 11y = 3. \quad (5)$$

(7) · 7 $\quad -7x - 49y = -63. \quad (8)$

(5) + (8), $\quad -60y = -60.$

$$y = 1.$$

Substituting in (7), $\quad x = 2.$

Substituting both these values in (1),

$$2 + 4 - 3z = 21.$$

Whence $\quad z = -5.$

Check. Substituting 2 for x, 1 for y, and -5 for z in (1), (2), and (3) respectively,

$$2 + 4 \cdot 1 - 3(-5) = 21, \text{ or } 21 = 21.$$
$$2 \cdot 2 - 5 \cdot 1 + (-5) = -6, \text{ or } -6 = -6.$$
$$5 \cdot 2 - 3 \cdot 1 + 2(-5) = -3, \text{ or } -3 = -3.$$

For the solution of a simultaneous system of equations in three unknowns we have the

RULE. *From an inspection of the coefficients decide which unknown is most easily eliminated.*

Using any two equations, eliminate that unknown.

With one of the equations just used and the third equation again eliminate the same unknown.

The last two operations give two equations in the same two unknowns. Solve these equations.

Substitute in the simplest of the original equations the two values found, and solve for the third unknown.

CHECK. *Substitute the values found in the original equations and simplify results.*

A system of four independent equations each of which contains the same four unknowns may be solved as follows:

Use the first and second equation, then the first and third, and lastly the first and fourth, and eliminate the same unknown each time. This gives a system of three equations in the same three unknowns, which can be solved by the rule given above.

EXERCISES

Solve for x, y, z, and w, and check the results:

1. $x + 3y - 5z = 2$,
   $2x - y - z = 1$,
   $3x + 5y - 7z = -10$.

2. $2x + 3y + 4z = -14$,
   $x - y + 3z = 0$,
   $5x + 2y + z = 14$.

3. $2x + y - 3z = 5,$
$x + 2y + z = 11,$
$3x - y + 2z = 4.$

4. $x + 2y + z = -1,$
$2x - y + z = -20,$
$-x - y - 5z = 13.$

5. $2x + 3y - 3z = -2,$
$4x - 2y + z = 9,$
$3x + y + 2z = 13.$

6. $2x + 3y + z = 14,$
$x - y + z = 4,$
$x + y - z = 0.$

7. $x + y + z = 1,$
$x + y - z = 2,$
$x - y + z = 3.$

8. $x + 2y + z = 1,$
$2x + y - z = 0,$
$x + 2y + z = 0.$

9. $2x + y + z = 7,$
$x - 3y + 2z = 13,$
$3x + 5y - z = -3.$

10. $2x - y + 5z = 0,$
$8x + 2y + z = 38,$
$x - 5y - z = 7.$

11. $2x + 3y + 6z = 6,$
$4x + y + z = 3,$
$x - 6y + 3z = \frac{1}{2}.$

12. $4x - 3y = z,$
$z = x + y,$
$2x = 3y + 1.$

13. $2x + y = 5 + z,$
$x - 6 = 2z,$
$3y + 2z = x.$

14. $x - 2y = 10,$
$3y + 1 = -4z,$
$5x - 18 = z.$

15. $x = 3z + 2,$
$y = x - 7\frac{1}{2},$
$z = 6y - 1.$

16. $\frac{a}{x} + \frac{b}{y} + \frac{c}{z} = 3,$
$\frac{a}{x} - \frac{b}{y} + \frac{2c}{z} = 2,$
$\frac{5a}{x} - \frac{2c}{z} = \frac{3b}{y}.$

17. $\frac{1}{2}(x+z-5) = y - z,$
$\frac{1}{2}(x+z-5) = 2x - 11,$
$2x - 11 = 9 - (x + 2z).$

18. $ax + by = 1,$
$by + cz = 1,$
$cz + ax = 1.$

19. $cy + bz = bc,$
$az + cx = ca,$
$bx + cy = ab.$

20. $2x + y + z + w = 1,$
$x - y - z + 2w = 4,$
$x + 2y - z - w = 0,$
$x - y + 2z - w = 1.$

21. $2x - y + z - w = 2,$
$3x + y - z - w = 0,$
$4x - 2y + z - w = -9,$
$2x - 3y - 2z + w = 7.$

22. $x + y + z = 6,$
$x + y + w = 7,$
$x + z + w = 8,$
$y + z + w = 9.$

PROBLEMS

1. The sum of two numbers is 139 and their difference is 39. Find the numbers.

2. The sum of two numbers is 17. Twice one less the other equals 7. What are the numbers?

3. The sum of two numbers is 13. Five times one less the other equals 29. What are the numbers?

4. There are three numbers such that the sum of the first and second is 67, the sum of the first and third is 79, and the sum of the second and third is 88. What are the numbers?

5. Forty-four tons of sand and gravel are required for the foundation of a building. It is found that the material can be hauled in a given time either by 12 trucks and 4 drays, or by 8 trucks and 10 drays. What is the capacity of a truck and of a dray?

6. What temperature on the Centigrade scale is one half that on the Fahrenheit scale?

HINT. Let the second equation be $C = \frac{1}{2}F$.

7. What temperature on the Fahrenheit scale is 3 times the equivalent temperature on the Centigrade scale?

8. Two men travel from San Francisco to Los Angeles by rail. It costs one of them three times as much for excess baggage as it costs the other. One pays \$19.10 in all, the other \$22.30. How much does each pay for his ticket?

9. If $cx + dy = 2$ is satisfied by $x = 2$ and $y = 3$, and also by $x = 3$ and $y = 4$, what values must c and d have?

10. If $2x + ay = m$ is satisfied when $x = 1$ and $y = -1$ and also when $x = 5$ and $y = 4$, what values must a and m have?

11. A boy rows 8 miles with the current in 1 hour 4 minutes, and returns against the current in $2\frac{2}{7}$ hours. At what rate would he row in still water? What is the rate of the current?

12. A and B can do a piece of work in s days, A and C in t days, and B and C can do it in r days. In how many days can each do the work?

13. A has a certain capital which is invested at a certain interest. B has \$5000 more capital than A, and invests it at 1 per cent more. His yearly income exceeds that of A by \$400. C has \$7500 more capital than A and invests it at 2 per cent more. His yearly income exceeds that of A by \$750. Find the capital of each and the rate at which it is invested.

14. A bag weighing 23 ounces contains two sizes of steel balls — ounce balls and $\frac{3}{5}$ ounce balls. There are 31 balls in all. Find the number of balls of each size.

15. If the length of a rectangle is increased by 9 feet and the breadth is decreased by 6 feet, the area is unchanged. If the length is decreased by 12 feet and the breadth is increased by 16 feet, the area is unchanged. Find the dimensions of the rectangle.

16. A man has \$3.55 in nickels and quarters. If he has 39 coins in all, how many has he of each?

17. A collection of 53 coins consists of nickels, dimes, and quarters. The value of the coins is \$8.40. Three times the number of quarters less twice the number of dimes is 7 less than 4 times the number of nickels. How many of each kind are there?

18. A man has \$12.00 in quarters, dimes, and nickels. He has as many quarters as he has dimes, and three times as many nickels as dimes. How many of each has he?

19. The difference between the second and third angles of a triangle is 5°; the difference between the second and first is 40°. How many degrees in each angle of the triangle?

20. Three men travelled from Los Angeles to Indianapolis and return. Two men shared the lower berth, and the third had an upper. The first two paid in all \$223.98, and the third paid \$119.64. The difference in price between a lower and an upper is \$5.10 each way. What was the cost of one round-trip ticket?

21. A man put \$12,000 at interest in three sums, the first at 5%, the second at 6%, and the third at $6\frac{1}{2}$%, receiving for the whole \$695 per year. The sum at 5% was half as much as the other two sums. Find each of the three sums.

REVIEW PROBLEMS

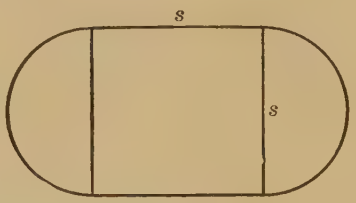
1. A boy has worked 30 problems. If he works 10 more and has 8 of them right, his average will be .9. How many problems has he already worked correctly?

2. A boy's grades in Geometry at the end of the first three quarters of the semester averaged .80. What must he make in the fourth quarter so that his average for the semester may be .84?

3. A rectangle $16' \times 4'$ has semi-circles added to each end. What is the perimeter of the figure after the semi-circles are attached?

4. What is the area of the figure in Problem 3?

5. A square of side s has semi-circles attached to opposite sides, as in the figure:



What is the perimeter? the area?

6. Four semi-circles are attached to a square. Obtain a formula for the area of the entire figure.

7. Find the area of the figure in Exercise 6 if $s = 7$ inches.

8. The formula for finding the area of a triangle in terms of its sides is $A = \sqrt{s(s-a)(s-b)(s-c)}$ (s = one half the sum of the sides a, b, and c). Find the area of a triangle whose sides are 7, 9, and 12 feet respectively.

9. $A = \frac{s^2}{4}\sqrt{3}$ is the formula for the area of an equilateral triangle. Find the area of an equilateral triangle whose side is 6.

10. A square of side s has equilateral triangles attached to each of the four sides. What is the area of the entire figure?

11. Using the formula of Problem 10, find the area of the figure if the side of the square is 8 inches.

CHAPTER VI

EXPONENTS

56. Proof of fundamental laws of exponents. The proofs for the four laws governing the use of positive integral exponents are as follows:

I. **Proof of the Law of Multiplication,**

$$x^a \cdot x^b = x^{a+b}.$$

If a and b are positive integers, we have

$$x^a = x \cdot x \cdot x \cdot x \cdots \text{to } a \text{ factors},$$

and

$$x^b = x \cdot x \cdot x \cdots \text{to } b \text{ factors}.$$

Hence $x^a \cdot x^b = (x \cdot x \cdot x \cdots \text{to } a \text{ factors}) \times (x \cdot x \cdot x \cdots \text{to } b \text{ factors})$

$= x \cdot x \cdot x \cdots \text{to } a + b \text{ factors}$

$= x^{a+b}$, by the definition of an exponent.

Law I may be stated more completely thus:

$$x^a \cdot x^b \cdot x^c \cdots = x^{a+b+c+\cdots}.$$

II. **Proof of the Law of Division,**

$$x^a \div x^b = x^{a-b}.$$

Again, if a and b are positive integers, we have

$$x^a \div x^b = \frac{x^a}{x^b} = \frac{x \cdot x \cdot x \cdots \text{to } a \text{ factors}}{x \cdot x \cdot x \cdots \text{to } b \text{ factors}}.$$

If b is less than a, the b factors of the denominator may be canceled with b factors of the numerator, leaving $a - b$ factors in the numerator.

Hence $$\frac{x^a}{x^b} = x^{a-b}.$$

If b is greater than a, a factors of the numerator may be canceled with a factors of the denominator, leaving $b - a$ factors in the denominator.

Hence $$\frac{x^a}{x^b} = \frac{1}{x^{b-a}}.$$

III. **Proof of the Law of Involution**, or raising to a power,

$$(x^a)^b = x^{ab}.$$

As before, if a and b are positive integers, we have

$$\begin{aligned}(x^a)^b &= x^a \cdot x^a \cdot x^a \cdots \text{to } b \text{ factors} \\ &= x^{a+a+a+\cdots \text{to } b \text{ terms of the exponent}} \\ &= x^{ab}.\end{aligned}$$

Law III includes the more general forms:

(*a*) $$(x^a y^b)^c = x^{ac} y^{bc}.$$

(*b*) $$[(x^a)^b]^{c\cdots} = x^{abc\cdots}.$$

When Laws I, II, and III were used in previous work in multiplication and division, we always assumed that a and b were positive integers and, in Law II, that a was greater than b. In the work on radicals (see "New First Course in Algebra," pp. 319–320) the meaning of an exponent was extended so as to include fractional exponents.

IV. **Proof of the Law of Evolution**, or extraction of roots,

$$\sqrt[b]{x^a} = x^{\frac{a}{b}}.$$

Assuming that Law I holds for fractional as well as integral exponents, we have

$$x^{\frac{a}{b}} \cdot x^{\frac{a}{b}} \cdot x^{\frac{a}{b}} \cdots \text{to } b \text{ factors} = x^{b\frac{a}{b}} = x^a.$$

Here the left member consists of b equal factors whose product is x^a. That is, each one of the factors must equal $\sqrt[b]{x^a}$.

This fact will be assumed without proof. We shall now explain the meaning which, according to these laws, must be given to a zero or to a negative exponent.

57. Meaning of zero as an exponent. From Law II we know that

$$y^5 \div y^5 = y^{5-5} = y^0.$$

But $$y^5 \div y^5 = \frac{y^5}{y^5} = 1.$$

Therefore $$y^0 = 1.$$

In general $$y^a \div y^a = y^{a-a} = y^0 = 1.$$

In other words, any expression (not equal to zero), raised to the zero power, becomes equal to 1. Thus, $(5)^0 = (\frac{1}{2})^0 = (-2)^0$ since each is equal to 1. Also, if x is not zero $(3x)^0 = 1$, and if y is not 2, the expression $(y^2 - 4y + 4)^0$ is equal to 1.

58. Meaning of a negative exponent. From Law II

$$x^2 \div x^4 = x^{2-4} = x^{-2}.$$

But $$x^2 \div x^4 = \frac{x^2}{x^4} = \frac{1}{x^2}.$$

Therefore $$\frac{1}{x^2} = x^{-2}.$$

Furthermore $3^{-3} = \frac{1}{3^3} = \frac{1}{27}$, and

$$m^{-\frac{3}{4}} = \frac{1}{m^{\frac{3}{4}}} = \frac{1}{\sqrt[4]{m^3}}.$$

The above facts may be expressed more generally as follows:

$$y^{-a} = \frac{1}{y^a}.$$

Therefore $\frac{1}{y^{-a}} = \frac{1}{\frac{1}{y^a}} = y^a.$

We also have

$$ay^{-b} = \frac{a}{y^b}, \text{ and } \frac{a}{y^{-b}} = ay^b.$$

We may therefore say: *In a fraction, any factor of the numerator may be omitted from the numerator and written as a factor of the denominator, and vice versa, provided that the sign of its exponent is changed.*

It follows that any expression involving negative exponents may be written as an expression involving only positive exponents. That is to say, negative exponents are not a mathematical necessity but merely a convenience. The extension of the laws of exponents which brings with it the zero exponent and the negative exponent is an illustration of what is called the Law of Permanence of Form.

It is to be understood that the part of the rule for multiplication (p. 6) and the rule for division (p. 9), which determines the exponents in the product or in the quotient, applies to all numbers, whether positive or negative, integral, fractional, or literal. Hence those rules need not be restated here.

ORAL EXERCISES

Perform the indicated operations and simplify results:

1. $x^{\frac{1}{3}} \cdot x^{\frac{2}{3}}$.
2. $x^{\frac{1}{5}} \cdot x^{\frac{3}{5}}$.
3. $x^{\frac{1}{3}} \cdot x^{\frac{1}{4}}$.
4. $y^{\frac{2}{3}} \cdot y^{\frac{1}{2}}$.
5. $y^{\frac{2}{3}} \div y^{\frac{1}{3}}$.
6. $x^{\frac{2}{5}} \cdot x^{\frac{3}{5}}$.
7. $r^2 \cdot r^{\frac{1}{2}}$.
8. $r^2 \div r^{\frac{1}{2}}$.
9. $x^{\frac{1}{3}} \cdot x$.
10. $m^{\frac{2}{3}} \cdot m^0$.
11. $m^{\frac{2}{3}} \div m^0$.
12. $y^b \cdot y^0$.
13. $p^2 \cdot p^{-1}$.
14. $x^3 \div x^{-1}$.
15. $x^{\frac{1}{3}} \cdot x^{-\frac{2}{3}}$.
16. $x^{\frac{1}{3}} \div x^{-\frac{2}{3}}$.
17. $y^a \div y^{-a}$.
18. $x^0 \cdot x^{\frac{1}{2}}$.
19. $x^0 \div x^{\frac{1}{2}}$.
20. $2\,m^0 \cdot (2\,m)^0$.
21. $12\,x^2 \cdot 2\,x^0$.
22. $4\,y^{\frac{1}{2}} \div 2\,y^0$.
23. $3\,x^2 \cdot x^0$.
24. $2\,y^{\frac{1}{2}} \cdot y^b$.
25. $r^{\frac{2}{3}} \div r^{\frac{5}{3}}$.
26. $r^{\frac{1}{2}} \cdot r \cdot r^{\frac{3}{2}}$.
27. $m^{\frac{1}{3}} \cdot m^{\frac{2}{3}} \cdot m^{\frac{1}{3}}$.
28. $y^0 \cdot y^4 \cdot y^{-\frac{1}{3}}$.
29. $(x^{\frac{1}{2}}+x^{\frac{1}{3}}) \cdot x^{\frac{1}{4}}$.
30. $m^{2a+1} \cdot m^{3a-2}$.
31. $x^{2+a} \cdot x^{a-2}$.
32. $a^2 \div a^{-2}$.
33. $y \cdot \sqrt{y}$.
34. $x^2 \div \sqrt[3]{x}$.
35. $\sqrt{m} \cdot \sqrt[4]{m}$.
36. $\sqrt[3]{y^5} \div \sqrt[3]{y^2}$.
37. $x \cdot \sqrt{x^3}$.
38. $\sqrt{m^0} \cdot \sqrt[3]{m^5}$.
39. $t^{b-2} \cdot t^{2-b}$.
40. $(x^3)^2 \cdot x^5$.
41. $(y^{\frac{1}{3}})^2 \cdot y^{\frac{1}{2}}$.
42. $x^{4b} \div \dfrac{1}{x^{3b}}$.
43. $m^2 \div \dfrac{1}{m^{-2}}$.
44. $y^{-2} \div \dfrac{1}{y^{-5}}$.
45. $(r^{\frac{1}{2}})^3 \cdot (r^3)^{-\frac{1}{2}}$.
46. $by^2 \div b^{-2}y^{-3}$.

Read the following with positive exponents and simplify the results:

47. r^{-2}.
48. $3\,m^{-4}$.
49. $2 \times y^{-3}$.
50. $8\,p^2r^{-4}$.
51. $3\,x^2y^{-3}z^{-2}$.
52. $\dfrac{z}{m^{-3}}$.
53. $\dfrac{2\,a}{3\,b^{-2}}$.
54. $\dfrac{3\,x^0}{a^{-1}b^{-3}}$.
55. $\dfrac{2\,m^{-2}p^0}{r^{-1}}$.
56. $\dfrac{2^{-2}(xy)^0}{5^{-3}x^2}$.
57. $\dfrac{2\,a^2bc^{-3}}{4\,a^0b^{-1}}$.

58. $\dfrac{15\,m^2y^{-1}}{5\,ym^{-2}}$.

59. $\dfrac{8^{-1}a^{-2}}{b^{-3}c^{-4}}$.

60. $\dfrac{3\,x^2y^{-5}z^3}{2^{-1}x^{-2}y^2z^0}$.

61. $\dfrac{2^{-3}m^{-2}p^{-4}}{2^{-4}m^{-3}p^{-5}}$.

62. $\dfrac{5\,x^{-2}}{y^{-5}z^2}$.

63. $3\,e^v \div \dfrac{4}{e^{-v}}$.

64. $\dfrac{a^{-2}}{a^{n+1}}$.

65. $(64\,x^{-6})^{-\frac{2}{3}}$.

66. $\dfrac{1}{(9\,a^{-5})^{-\frac{1}{10}}}$.

Simplify and arrange terms so that the exponents occur in the descending order:

67. $a^3 + a^{-2} + 5\,a^{\frac{7}{2}} - 4\,a^0$.

68. $2\,x^{-\frac{1}{4}} + 3\,x^{-3} - x^{\frac{1}{2}} + 2\,x^3$.

69. $m^{\frac{1}{3}} + 2\,m^{-2} - 1 + m^{-\frac{3}{2}} - m^2$.

70. $x + 2\,x^{-\frac{1}{2}} - 3\,x^{\frac{4}{3}} + 2\,x^0 - 5\,x^{\frac{1}{2}}$.

71. $y^2 + \dfrac{1}{y^2} + y^{-\frac{1}{2}} - y^{\frac{1}{2}} + \dfrac{1}{y^0}$.

72. $m^{\frac{2}{3}} + \dfrac{1}{m^2} - 3 - \dfrac{1}{m^{-3}} + \dfrac{1}{m}$.

EXERCISES

Perform the indicated multiplications and write results as integral expressions:

1. $(x + x^{\frac{2}{3}} - x^{\frac{1}{2}})3\,x^{\frac{1}{3}}$.
2. $(x^3 - y^3)x^{-6}y^{-6}$.
3. $(x^{\frac{1}{2}} + y^{\frac{1}{2}})(x^{\frac{1}{2}} + y^{\frac{1}{2}})$.
4. $(m^{\frac{1}{2}} - p^{\frac{1}{2}})(m^{\frac{1}{2}} + p^{\frac{1}{2}})$.
5. $(x^{\frac{1}{3}} - y^{\frac{1}{3}})(x^{\frac{1}{3}} + y^{\frac{1}{3}})$.
6. $(b^{-4} + 2)(b^{-4} + 3)$.
7. $(y^{3b} - 4\,y^{2b})(y^3 - 3\,y)$.
8. $(m^{-2} + m)^2$.
9. $(x^4 - 3\,x^{-1})^3$.
10. $(4\,z + 2\,z^{-1} - 3\,z^{-2})^2$.
11. $(m^x + m^{-x})^2$.
12. $(m^x - m^{-x})^2$.

13. $(a^9 - 1)^3$.

14. $(x^{-1} + y^{-2})\left(\frac{1}{x} - \frac{1}{y^2}\right)$.

15. $(a^{2y} - 4 + a^{-2y})^2$.

16. $(4\,m^{-\frac{1}{2}} - 3\,m^{\frac{1}{2}})^2$.

17. $(2\,x - 5\,x^{-1} + x^{-2})^2$.

18. $(e^{3x} - 2\,e^{-x})^4$.

19. $(x + x^{\frac{1}{2}}y^{\frac{1}{2}} + y)(x - x^{\frac{1}{2}}y^{\frac{1}{2}} + y)$.

20. $(p^{\frac{1}{2}} + q^{\frac{1}{2}} + p^{\frac{1}{4}}q^{\frac{1}{4}})(q^{\frac{1}{2}} - p^{\frac{1}{4}}q^{\frac{1}{4}} + p^{\frac{1}{2}})$.

21. $(a + a^{\frac{1}{2}}b^{-\frac{1}{2}} + b^{-1})(a - a^{\frac{1}{2}}b^{-\frac{1}{2}} - b^{-1})$.

22. $(\sqrt{a^3} + 3\sqrt[3]{a^4})(\sqrt{a^3} + 3\sqrt[3]{a^4})$.

23. $(2\sqrt{x^{-1}} - 5\sqrt[4]{y^3} + a\sqrt[3]{x^{-4}})^2$.

24. $(x^{\frac{1}{3}} - y^{\frac{2}{3}})(\sqrt[3]{x} - \sqrt[3]{y^2})$.

25. $\left(\sqrt{m} - \frac{\sqrt{n}}{p}\right)\left(\sqrt{m} + \frac{\sqrt{n}}{p}\right)$.

EXERCISES

Perform the indicated division and write results as integral expressions:

1. $a^2 \div a^7$.

2. $x^2 \div x^{\frac{1}{2}}$.

3. $y^{\frac{1}{3}} \div y^3$.

4. $ab^{\frac{3}{4}} \div a^{\frac{1}{2}}b^{\frac{1}{4}}$.

5. $\dfrac{xy + x^3y^2}{x^2y^{\frac{1}{3}}}$.

6. $(m^b - 3\,m^{5b+2} + 2\,m^{b-1}) \div m^{2b-1}$.

7. $(8\,x^{y-4} + 2\,x^{5+y} - 6\,x^{3-y}) \div 2\,x^{y-3}$.

8. $(a - b) \div (a^{\frac{1}{2}} + b^{\frac{1}{2}})$.

9. $(x - y) \div (x^{\frac{1}{2}} - y^{\frac{1}{2}})$.

10. $(x + y) \div (x^{\frac{1}{3}} + y^{\frac{1}{3}})$.

11. $(9\,a - 16\,b) \div (3\,a^{\frac{1}{2}} + 4\,b^{\frac{1}{2}})$.

12. $(x^{\frac{3}{4}} + y^{\frac{1}{4}}) \div (\sqrt[4]{x} + \sqrt[12]{y})$.

13. $(a^4 + a^2b^{-1} + b^{-2}) \div (a^2 - ab^{-\frac{1}{2}} + b^{-1})$.

14. $(e^{2y} + e^{-2y} + 2) \div (e^y + e^{-y})$.

15. $(2\,a^{3m} + 3\,a^{2m} + 2\,a^m - 5\,a^{-2m} - 2) \div (a^m + a^{-m})$.

16. $(e^{-3x} + e^{3x} - e^{-x} - e^x) \div (e^{-x} - e^x)$.

17. $(a + 9\,a^{\frac{2}{3}}b^{\frac{2}{3}} + 27\,a^{\frac{1}{3}}b^{\frac{4}{3}} + 27\,b^2) \div (a^{\frac{1}{3}} + 3\,b^{\frac{2}{3}})$.

18. $(a^4 + 6\,a^{-2} + 6\,a^2 + a^{-4} + 11) \div (a^{-2} + a^2 + 3)$.

19. $(9\,x^{5n-4} - x^{3n-2} + 2\,x^{2n-1}) \div (2\,x^{n-1} + 3\,x^{2n-2})$.

20. $(119 + 44\,x^a + 21\,x^{-a} + 4\,x^{2a} - 6\,x^{-2a}) \div (9 + 3x^{-a} + 2x^a)$.

Divide:

21. $a^{\frac{4}{3}} - b^{\frac{4}{3}} + a^{\frac{1}{3}}b - ab^{\frac{1}{3}}$ by $a^{\frac{1}{3}} - b^{\frac{1}{3}}$.

22. $3\,r^{-5} + r^3 - 4\,r^{-3}$ by $2\,r^{-1} + r + 3\,r^{-3}$.

23. $x^{\frac{3}{5}} - m^{\frac{3}{5}}$ by $[(x^{\frac{1}{5}} - m^{\frac{1}{5}}) \div (x^{\frac{1}{10}} + m^{\frac{1}{10}})]$.

24. $40 - 12\,a - 10\,a^{-1} - 4\,a^{-\frac{1}{2}} - 14\,a^{\frac{1}{2}}$ by $2\,a^{\frac{1}{2}} + 3 - 5\,a^{-\frac{1}{2}}$.

25. $93 - 10(x^{-2a} + x^{2a}) - 24(x^{-a} + x^a)$ by $2\,x^a - 5\,x^{-a} + 8$.

NOTE. To us, who use the notation of exponents every day, it seems so simple and natural a method of expressing the product of several factors that it is difficult to understand why such a long time was necessary to develop it. But here, as in many other instances, it required a great man to discover what to us seems the most obvious relation. The man who brought the notation of exponents to its modern form was John Wallis (1616–1703), an Englishman.

Though the idea of using negative and fractional exponents had occurred to writers before Wallis, it was he who showed their naturalness, and who introduced them permanently. He also was the first to use the ordinary sign ∞ to denote infinity.

MISCELLANEOUS EXERCISES

Find the numeric value of the following:

1. 2^{-1}.
2. 3^{-3}.
3. 5^{-2}.
4. $2^{-3} \cdot 4^{0}$.
5. $3^{-4} \cdot 2^{-1}$.
6. $2^{-5} \cdot 5^{2} \cdot 0$.
7. $(\frac{1}{3})^{-1}$.
8. $(\frac{1}{2})^{-2}$.
9. $(\frac{2}{3})^{-3} \cdot (2)^{0}$.
10. $\dfrac{2^{-3} \cdot 5^{-2}}{10^{-2}}$.
11. $\dfrac{1}{243^{-\frac{2}{5}}}$.
12. $0^{2} \cdot 3^{-0}$.
13. $9^{-\frac{3}{2}}$.
14. $27^{-\frac{2}{3}}$.
15. $81^{-\frac{1}{4}}$.
16. $16^{-\frac{3}{4}}$.
17. $25^{-\frac{3}{2}}$.
18. $16^{.5}$.
19. $81^{1.25}$.
20. $(-8)^{-\frac{1}{3}}$.
21. $(9)^{-\frac{3}{2}}$.
22. $(2)^{-4}$.
23. $-64^{-\frac{5}{6}}$.
24. $\sqrt[4]{16^{-3}}$.
25. $-\sqrt[3]{-27^{2}}$.
26. $(\frac{1}{2})^{-3} \cdot (\frac{1}{3})^{0} \cdot (\frac{1}{4})^{\frac{3}{2}}$.
27. $(.25)^{\frac{1}{2}}$.
28. $(.09)^{-\frac{3}{2}}$.
29. $(.0016)^{\frac{3}{4}}$.
30. $\dfrac{\sqrt[4]{2^{-4}} \cdot \sqrt{2^{-2}}}{2^{-2}}$.

Write with positive exponents and simplify the results:

31. $\dfrac{3}{m^{-2} + b^{-2}}$.
32. $\dfrac{3\,a^{2}b}{a^{-3} - b^{-2}}$.
33. $\dfrac{a^{-2}}{b^{-2} + c^{-2}}$.
34. $\dfrac{x^{-3} - 8\,m^{-3}}{x^{-1} - 2\,m^{-1}}$.
35. $\dfrac{x^{-2} - y^{-2}}{x^{-1} + y^{-1}}$.
36. $\dfrac{m^{-3} + 27\,a^{-3}}{m^{-1} + 3\,a^{-1}}$.

Write without a denominator and simplify the results:

37. $\dfrac{3\,mb}{x^{2}}$.
38. $\dfrac{11\,x^{2}}{y}$.
39. $\dfrac{2\,r}{3\,s^{-5}b^{2}}$.
40. $\dfrac{3\,a^{-1}b}{3^{-1}ab^{3}}$.
41. $\dfrac{15\,m^{-2}n^{-1}}{5\,m^{2}n}$.
42. $\dfrac{2\,ab^{-2}}{(a-b)^{2}}$.

43. $\dfrac{6\,m^3(n-1)}{(n-1)^{-\frac{1}{3}}}$.

44. $\dfrac{2}{6\,a^2(a+b^2)^{-\frac{1}{2}}}$.

45. $\dfrac{2(x+y)^2}{3\,m(x-y)^{-\frac{3}{2}}}$.

46. $\dfrac{5(a+b^2)^{-3}}{2(a^{-2}+b)^2}$.

47. $\dfrac{(18\,x^{-y})\cdot(y^{-2x})}{36\,x^{-2}y^{-3x}}$.

48. $\dfrac{p^{-2}q^{-1}}{p^2q^{-4}(p^2-q^{-2})^{\frac{1}{2}}}$.

Simplify:

49. $(16\,a^{12}b^8)^{-\frac{1}{4}}$.

50. $(3\,a^{10})^0\cdot 8\cdot 16^{-\frac{1}{2}}$.

51. $(p^2q)^{-3}(p-q)$.

52. $(m^2)^{3b}\cdot m^{3b}$.

53. $(m^{2x})^{3y}\cdot(m^{2y})^{3x}$.

54. $(x^{a+2})^2\cdot(x^{2-a})^2$.

55. $(y^2)^{a+b}(y^3)^{a-b}$.

56. $(p^2q)^m(pq^2)^{3m}$.

57. $(a^{-2}+a^2)a^2$.

58. $(x^2-3\,xy^{-1})x^{-2}$.

59. $2^3\cdot 8^2 = 2^?$.

60. $2^a\cdot 4^b = 2^?$.

61. $\dfrac{3^x\cdot 9^{2x}}{27^{4x}} = 3^?$.

62. $2^m\cdot 4^{3m}\div 2^{2n} = 2^?$.

63. $\dfrac{3^{a+2}}{9^a(3^{3a-1})^a}\div\dfrac{9^{a+1}}{(27^{2a-3})^{a+1}}+3 = ?$

Solve for x:

64. $x^{\frac{1}{2}} = 3$.

65. $x^{\frac{3}{2}} = 4$.

66. $x^{-\frac{1}{2}} = 3$.

67. $x^{-\frac{1}{4}} = 2$.

68. $x^{\frac{2}{3}} = 9$.

69. $x^{-\frac{3}{4}} = 8$.

70. $x^{\frac{1}{4}} = -2$.

71. $x^{\frac{4}{3}} = 81$.

72. $x^{-\frac{2}{3}} = \frac{1}{4}$.

73. $x^{-\frac{3}{2}} = 216$.

74. $\frac{1}{4}\,x^{-\frac{2}{3}} = 1$.

75. $(x^{-\frac{1}{2}})^{-2} = 2$.

76. $(a^{\frac{1}{2}}x^{\frac{1}{3}})^{-6} = 9$.

77. $\dfrac{\sqrt[4]{x^{\frac{2}{3}}}}{\sqrt[4]{x^{\frac{1}{3}}}} = \dfrac{\sqrt[3]{4}}{\sqrt[3]{9}}$.

CHAPTER VII

SQUARE ROOT

59. Square root. The square root of any number is one of the two equal factors whose product is the number.

From the law of signs in multiplication it follows that

Every positive number or algebraic expression has two square roots which have the same absolute value but opposite signs.

Thus $\sqrt{a} \cdot \sqrt{a} = (-\sqrt{a})(-\sqrt{a}) = a.$

60. Square root of a monomial. For extracting the square roots of any monomial we have the

RULE. *Write the square root of the numeric coefficient preceded by the double sign $\pm$ and followed by all the letters of the monomial, giving to each letter an exponent equal to one half its exponent in the monomial.*

A rule similar to the preceding one holds for the fourth root, the sixth root, and other even roots.

61. Cube root. The cube root of any number is one of the three equal factors whose product is the number.

For extracting the cube root of a monomial we have the

RULE. *Write the cube root of the numeric coefficient preceded by the sign of the monomial followed by all the letters of the monomial, giving to each letter an exponent equal to one third of its exponent in the monomial.*

A rule similar to the preceding one holds for the fifth root, the seventh root, and other odd roots.

62. Principal root. For a given index (see section 66) the **principal root** of a number is its one real root if it has but one, or its positive real root if it has two real roots.

Then the principal square root of 9 is $+3$; the principal fourth root of 16 is $+2$, not -2. The square root of -4 or -9 is *not* a real number; a negative number has no principal square root.

The principal cube root of 8 is 2, of -27 is -3. The principal fifth root of 32 is $+2$, of -32 is -2.

Every number has more than one root of given odd index. The number 8, for example, has two other cube roots besides its principal cube root 2. What they are and how they are obtained will be made clear in the chapter on Imaginaries, where the consideration of the square roots of negative numbers is taken up.

Only the principal odd root of a number will be considered in the following exercises.

ORAL EXERCISES

Find the principal square root of the following:

1. 9.
2. 36.
3. x^2.
4. $25\,b^4$.
5. $49\,m^6$.
6. $36\,r^4$.
7. $4\,p^{12}$.
8. $16\,x^8$.
9. $\frac{1}{9}$.
10. $\frac{1}{x^2}$.
11. $\frac{4}{m^4}$.
12. $\frac{a^2}{b^2}$.
13. a^{-2}.
14. $4\,x^{-4}$.
15. $25\,a^2b^{-2}$.
16. $100\,p^0$.
17. $x^{\frac{2}{3}}$.
18. $m^{\frac{4}{5}}$.

Find the principal cube root of the following:

19. 8.
20. 27.
21. 125.
22. $8\,x^3$.
23. $64\,y^6$.
24. $8\,a^{12}$.
25. -8.
26. -64.
27. -27.
28. $-125\,x^6$.
29. $-8\,b^3$.
30. $-64\,z^6$.
31. $216\,x^3$.
32. $343\,x^6$.
33. $512\,m^3$.
34. $-64\,x^{\frac{3}{2}}$.
35. $-8\,a^{-3}$.
36. $-125\,m^{-12}$.

Find the principal fourth root of the following:

37. 81.
38. 16.
39. 625.
40. r^4.
41. x^8.
42. m^{12}.
43. $16\,a^4$.
44. $625\,b^{-4}$.
45. $81\,c^{-12}$.
46. $m^{-\frac{4}{3}}$.
47. $r^{-\frac{8}{5}}$.
48. $y^{-\frac{4}{7}}$.

Give the principal root and one other root for the following:

49. The fourth root of 16; of x^8; of x^{-12}.

50. The sixth root of 64; of a^{12}; of a^{-12}.

51. What is the sign of the principal *odd* root of a positive number? the principal odd root of a negative number?

52. What is the sign of the principal even root of a positive number?

53. State a rule for extracting the fourth root of a monomial.

54. State a rule for extracting the fifth root of a monomial.

55. Can one obtain the fifth root of a monomial by extracting the square root of its cube root? by extracting the cube root of its square root? Explain.

63. Square root of polynomials. Extracting the square root of a number is essentially an undoing of the work of multiplication. The square of any polynomial may be represented by

$$(h + t + u)^2 = h^2 + 2\,ht + t^2 + 2\,hu + 2\,tu + u^2$$
$$= h^2 + (2\,h + t)t + (2\,h + 2\,t + u)u.$$

A little study of this last form and a comparison with the example which follows will make clear the reason for each step of the process.

EXAMPLE

Extract the square roots of $h^2 + 2\,ht + t^2 + 2\,hu + 2\,tu + u^2$.

Solution.

$h + t + u$

$h^2 + 2\,ht + t^2 + 2\,hu + 2\,tu + u^2$

$h^2 = (h)^2$

First trial divisor, $2\,h$ | $+2\,ht + t^2$

First complete divisor, $2\,h + t$ | $+2\,ht + t^2 = (2\,h + t)t$

Second trial divisor, $2\,h + 2\,t$ | $+2\,hu + 2\,tu + u^2$

Second complete divisor, $2\,h + 2\,t + u$ | $+2\,hu + 2\,tu + u^2 = (2\,h + 2\,t + u)u$

Therefore the required square roots are $\pm\,(h + t + u)$.

EXERCISES

Extract the square roots of the following:

1. $9\,x^4 - 12\,x^3 + 16\,x^2 - 8\,x + 4.$
2. $4\,r^6 - 12\,r^5 + 17\,r^4 - 32\,r^3 + 34\,r^2 - 20\,r + 25.$
3. $m^6 + 6\,m^4 + 4\,m^3 + 9\,m^2 + 4 + 12\,m.$
4. $16\,p^6 + 24\,p^5 - 80\,p^3 + 9\,p^4 + 100 - 60\,p^2.$
5. $4\,t^4 - 20\,a^2t^2 + 12\,t^2 + 25\,a^4 + 9 - 30\,a^2.$
6. $9\,x^{-4} - 12 + 4\,x^4.$
7. $16\,a^{-4} - 24\,a^{-2} - 30\,a^2 + 25\,a^4 + 49.$

8. $625 + 9\,a^2 - 6\,a^{\frac{3}{2}} - 149\,a + 50\,a^{\frac{1}{2}}$.

9. $81\,x^3 + 36\,x^{\frac{3}{2}} - 54\,x + 4 - 12\,x^{-\frac{1}{2}} + 9\,x^{-1}$.

10. $4\,x^3 - 8\,x^{\frac{3}{2}} + 20\,x + 4 - 20\,x^{-\frac{1}{2}} + 25\,x^{-1}$.

11. $144\,r^{16} + 9\,r^{-10} + 100\,r^{-4} - 72\,r^3 - 240\,r^6 + 60\,r^{-7}$.

12. $\dfrac{9\,x^2}{4\,y^2} + \dfrac{4\,y^2}{9\,x^2} - \dfrac{15\,x}{y} + 27 - \dfrac{20\,y}{3\,x}$.

HINT. Reduce the expression to a single fraction and find the square root of the numerator and denominator separately.

13. $4\,x^4 - 12\,x^3 + \frac{1}{4} - 3\,x + 11\,x^2$.

14. $4\,x^6 - 4\,x - \frac{1}{2}\,x^{-2} + x^3 + \frac{1}{16} + x^{-4}$.

15. $\dfrac{9\,a^4}{25} - \dfrac{4\,a^3}{5} - \dfrac{10\,a}{3} + \dfrac{25}{4} + \dfrac{31\,a^2}{9}$.

16. $\dfrac{a^2}{4} + \dfrac{ab}{3} + \dfrac{b^2}{9} - a + 1 - \dfrac{2\,b}{3}$.

17. $\frac{4}{9}\,a^6 - \frac{2}{3}\,a^4 + \frac{8}{3}\,a^3 + \frac{1}{4}\,a^2 + 4 - 2\,a$.

18. $4\,m^2 + \dfrac{1}{m^2} - \dfrac{10}{m} + 21 + 20\,m$.

19. $\dfrac{x^2}{4} + \dfrac{4}{x^2} + 3 - x - \dfrac{4}{x}$.

20. $2 + \dfrac{a^2}{b^2} + \dfrac{4\,a}{b} + \dfrac{b^2}{a^2} - \dfrac{4\,b}{a}$.

21. $\dfrac{c^2}{d^2} + c^2d^2 + \dfrac{d^2}{c^2} + 2 - 2\,c^2 - 2\,d^2$.

22. $\dfrac{1}{4\,y^2} - \dfrac{3}{y} + \dfrac{2}{3\,x} + 9 - \dfrac{4\,y}{x} + \dfrac{4\,y^2}{9\,x^2}$.

23. $\dfrac{4\,a^2}{9\,b^2} + 1\dfrac{8}{9} + \dfrac{4\,a}{3\,b} + \dfrac{4\,b^2}{9\,a^2} + \dfrac{4\,b}{3\,a}$.

Find the first four terms in the square root of the following:

24. $3 + 5x$.
25. $x + 2$.
26. $x^3 + x$.
27. $\frac{1}{25} + m^2$.

64. Square root of arithmetic numbers. The abbreviated process of extracting the square roots of an arithmetic number is as follows:

$$
\begin{array}{rl}
 & 2\ 2\ 1\ 0.\ 1\ 9+ \\
 & \overline{4'88'49'63.00'00} \\
 & 4 \\
4\,2 & \big|\ 88 \\
 & \big|\ 84 \\
\hline
44\,1 & \quad 4\,49 \\
 & \quad 4\,41 \\
\hline
4420\,1 & \qquad 8\,63\,00 \\
 & \qquad 4\,42\,01 \\
\hline
44202\ 9 & \qquad 4\,20\,99\,00 \\
 & \qquad 3\,97\,82\,61 \\
\hline
 & \qquad\quad 23\,16\,39
\end{array}
$$

Therefore the square roots of 4,884,963 correct to the first decimal place are $\pm$ 2210.2.

It follows from the preceding example that the work of extracting the positive square root of a number may be a never-ending process. The number 4,884,963 has no exact square root, and no matter how far the work is carried, there is no final digit. As the work stands we know that the required root lies between 2210.1 and 2210.2 and that it is nearer 2210.2.

The method just illustrated for extracting the positive square root of a number is the one commonly used. For it we have the

RULE. *Begin at the decimal point and point off as many groups of two digits each as possible: to the left if the number is an integer, to the right if it is a decimal, to both the left and the right if the number is part integral and part decimal.*

Find the greatest integer whose square is equal to or less than the left-hand group and write this integer for the first digit of the root, directly over the group of digits used in determining it.

Square the first digit of the root, subtract its square from the first group and annex the second group to the remainder.

Double the part of the root already found for a trial divisor, divide it into the remainder (omitting from the latter the right-hand digit), and write the integral part of the quotient as the next digit of the root and directly over the group of digits used in determining it.

Annex the root digit just found to the trial divisor to make the complete divisor, multiply the complete divisor by this root digit, subtract the result from the dividend, and annex to the remainder the next group, for a new dividend.

Double the part of the root already found, for a new trial divisor, and proceed as before until the desired number of digits of the root have been found.

After extracting the square root of a number involving decimals, point off one decimal place in the root for every decimal group in the number.

CHECK. *If the root is exact, square it. The result should be the original number. If the root is inexact, square it and add to this result the remainder. The sum should be the original number.*

EXERCISES

Find the positive square roots of the following:

1. 2,401.	5. 67.24.	9. 7,387,524.
2. 5,184.	6. 1.5876.	10. 3,519,376.
3. 18,496.	7. 231.04.	11. 7.7284.
4. 59,049.	8. 585,225.	12. .00018496.

Find correct to three decimal places the positive square roots of the following:

13. 6. 15. .08754. 17. $\frac{6}{11}$. 19. $\frac{12}{7}$. 21. $14\frac{3}{8}$.

14. .85. 16. .998765. 18. $3\frac{2}{3}$. 20. $\frac{5}{7}$. 22. $78\frac{1}{6}$.

23. Find the hypotenuse of a right triangle whose sides are 15 feet and 112 feet.

24. A baseball diamond is 85 feet on a side. What is the distance from first base to third base, to the nearest hundredth of a foot?

25. A right triangle has a hypotenuse 75 feet and one side 54 feet. Find the remaining side to the nearest hundredth of a foot.

26. The hypotenuse and one side of a right triangle are respectively 3569 feet and 2576 feet. Find the remaining side to the nearest tenth of a foot.

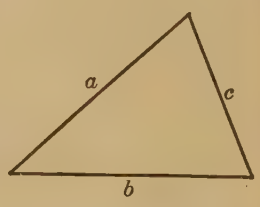
27. The side of an equilateral triangle is 21 inches. Find the altitude to the nearest tenth of an inch.

28. Find the side of an equilateral triangle whose altitude is 24 inches, to the nearest hundredth of an inch.

Fact from Geometry. Area of triangle

$A = \sqrt{s(s-a)(s-b)(s-c)}$

where a, b, and c = the sides of the triangle and $s = \frac{1}{2}$ the perimeter of the triangle.



29. A triangle has sides 12 inches, 14 inches, and 16 inches. What is its area to the nearest hundredth of a square inch?

30. By the method of Exercise 29 find to the nearest hundredth of a square inch the area of a triangle, each side of which is 16 inches.

31. Find correct to two decimal places the sum of all the diagonal lines that can be drawn on the faces of a cube whose side is 21 inches.

32. Find to the nearest thousandth of an inch the radius of a circle whose area is 432 square inches.

33. A room is 18 feet by 21 feet by 12 feet. What is the diagonal of the room to the nearest tenth of a foot?

34. Find to the nearest thousandth of an inch the diagonal of a cube 2 inches on a side.

35. A cube has an outside area of 17,592 square inches. Find the volume to the nearest tenth of a cubic inch.

Using the table on page 411 for squares and square roots:

36. Find the square roots of 8; 28; 45; 84; 57. Compare the results found by computing the required square roots to three decimal places.

37. Find the diagonals of the squares whose sides are 3; 7; 6.

38. Find the hypotenuse of the right triangles whose sides are 5 feet and 7 feet; 3 inches and 8 inches; 2 rods and 9 rods.

CHAPTER VIII

RADICALS

65. Radical. A **radical** is an indicated root of an algebraic or arithmetic expression.

Thus $\sqrt{4}$, $\sqrt{7}$, $\sqrt[3]{3\,y}$, and $\sqrt{r^2 + 3\,r - 1}$ are radicals.

66. Index. The small figure like the 3 in $\sqrt[3]{5}$ is called the **index** of the radical.

The index determines the **order** of the radical and indicates the root to be extracted.

For the square root the index is usually omitted. Thus $\sqrt{3}$ and $\sqrt{22}$ mean $\sqrt[2]{3}$ and $\sqrt[2]{22}$ respectively, and are of the second order.

67. Radicand. The **radicand** is the number or expression under the radical sign. In $\sqrt{8}$ and $\sqrt[3]{5\,am}$ the respective radicands are 8 and $5\,am$.

68. Fractional exponents. Radical expressions may be written in either of two ways: with radical signs or with fractional exponents.

Thus $\sqrt{7}$ and $7^{\frac{1}{2}}$ have the same meaning, and $\sqrt[3]{c^2}$ equals $c^{\frac{2}{3}}$, etc.

69. Rational numbers. A **rational number** is a positive or a negative integer or any number which can be expressed as the quotient, or ratio, of two such integers.

Thus 8, -4, $\frac{3}{5}$, or 6.713 are rational numbers.

70. Irrational numbers. Any real number which is not rational is **irrational**. (See section 72.)

If a number under a radical sign is such that the root indicated cannot be exactly obtained, the radical represents an **irrational** number.

For example, $\sqrt{5}$ and $\sqrt[3]{7}$ are irrational. Approximate values for these are given on page 411.

A repeating decimal, though endless, is not an irrational number, for any repeating decimal can be expressed as a common fraction, and is therefore rational.

Thus the repeating decimal .272727··· is not irrational, as it exactly equals $\frac{3}{11}$. Similarly, .2857142857142··· exactly equals $\frac{2}{7}$, etc.

NOTE. The number $\frac{1}{7}$ reduced to a decimal repeats the digits in groups of six each, and the mere fact that a decimal does so repeat is proof that it is a rational number. On the other hand, the number π is known to be irrational, and its value has been computed to 707 decimals, showing no repetition. The fact that it does not repeat in 700 digits is, however, no proof that π is irrational, for decimals with even more than that many digits do repeat. For example, the fraction $\frac{10000}{7699}$ equals the decimal 1.29^{+}, which repeats in groups of 7698 digits each.

71. Imaginary. An indicated *square* root of a *negative* number is called an **imaginary** number.

Thus $\sqrt{-9}$, $\sqrt{-5}$, and $\sqrt{-15}$ are imaginary numbers. And $5+\sqrt{-1}$ is also imaginary, though, as will be seen later (Chapter XI), such numbers are usually called complex numbers.

72. Classification of numbers. All the numbers of algebra then may be placed in one or the other of two classes: **real** numbers and **imaginary** numbers.

Real numbers, as we have seen, are of two kinds, **rational** numbers and **irrational** numbers.

73. Surd. A **surd** is an irrational number in which the radicand is rational.

Thus $\sqrt{2}$, $\sqrt[3]{5}$, etc., are surds. But $\sqrt{5+\sqrt{2}}$ and $\sqrt{\pi}$ are not surds.

74. The algebraic sign of a radical. The square root of 9 is both $+3$ and -3. The symbol $\sqrt{9}$, however, signifies only $+3$, the principal root (section 62). Similarly, $\sqrt[4]{81}$ is $+3$ and $\sqrt[6]{64}$ is $+2$. But $-\sqrt{9}$ is -3, and $-\sqrt[4]{16}$ is -2. The symbol $\pm\sqrt{36}$ denotes both $+6$ and -6. Further, $+\sqrt[3]{27}=+3$, $-\sqrt[3]{27}=-3$, and $-\sqrt[3]{-27}=+3$.

The foregoing remarks apply also to fractional exponents. Thus $4^{\frac{1}{2}}=+2$ only, and $81^{\frac{1}{4}}=+3$ only. It should be noted that these statements really define the meaning of such symbols as $\sqrt{\ }$, $\sqrt[4]{\ }$, $\sqrt[6]{\ }$. Such an understanding as this avoids all the ambiguity which would arise if $\sqrt{16}$ meant both $+4$ and -4. The distinctions here made are especially needed in checking irrational equations (Chapter X).

ORAL EXERCISES

Find the numerical value of the following:

1. $\sqrt{9}$.
2. $-\sqrt{4}$.
3. $\sqrt{25}$.
4. $\sqrt[3]{8}$.
5. $\sqrt[3]{-27}$.
6. $-\sqrt[4]{625}$.
7. $-\sqrt[4]{16}$.
8. $\sqrt[3]{-8}$.
9. $\sqrt[5]{243}$.
10. $-\sqrt[7]{-128}$.
11. $-\sqrt[6]{64}$.
12. $-\sqrt[3]{-125}$.
13. $4^{\frac{1}{2}}$.
14. $27^{\frac{1}{3}}$.
15. $25^{\frac{1}{2}}$.
16. $9^{\frac{3}{2}}$.
17. $81^{\frac{1}{4}}$.
18. $8^{\frac{2}{3}}$.
19. $16^{\frac{5}{4}}$.
20. $(\frac{1}{4})^{\frac{1}{2}}$.
21. $(\frac{1}{25})^{\frac{1}{2}}$.
22. $(-125)^{\frac{1}{3}}$.
23. $(36)^{\frac{3}{2}}$.
24. $(125)^{\frac{4}{3}}$.

Read as if expressed in radical form:

25. $x^{\frac{2}{3}}$.
26. $y^{\frac{3}{5}}$.
27. $(xy)^{\frac{3}{2}}$.
28. $(5\,m)^{\frac{1}{2}}$.
29. $5\,m^{\frac{1}{3}}$.
30. $2\,mx^{\frac{1}{4}}$.
31. $9\,ap^{\frac{3}{4}}$.
32. $3\,m^{\frac{1}{2}}p^{\frac{1}{3}}$.
33. $6\,ab^{\frac{2}{3}}$.
34. $2\,p^2q^{\frac{3}{4}}$.
35. $x(x-1)^{\frac{1}{3}}$.
36. $3\,b(2\,a-5)^{\frac{2}{5}}$.
37. $y^{\frac{1}{a}}$.
38. $3^{\frac{1}{2}}z^{\frac{1}{b}}$.
39. $x^{\frac{a}{b}}y^{\frac{3b}{a}}$.
40. $x^{\frac{m}{2}}y^{\frac{3m}{4}}$.

Read as if expressed with fractional exponents:

41. $\sqrt{x^3}$.
42. $\sqrt{ab^5}$.
43. $\sqrt[3]{x^4}$.
44. $\sqrt[3]{m^2x^7}$.
45. $3\sqrt{m}$.
46. $\sqrt[3]{2\,z^2}$.
47. $\sqrt[3]{(m+n)^4}$.
48. $\sqrt{(2\,x+y)^3}$.
49. $\sqrt[3]{m^4(x-2)}$.

50. What are the two square roots of 25?

51. What are two fourth roots of 81? of 256? of 625?

52. What are the numeric values of $\sqrt[4]{81}$? of $\sqrt[4]{256}$? of $\sqrt[4]{625}$?

53. What are two sixth roots of 729?

54. What is the distinction between a rational number and an irrational one?

55. Which of the numbers 4, $\frac{1}{2}$, 365, $\sqrt{9}$, $\sqrt{5}$, and π are rational? Which are irrational?

56. Give a geometric illustration of an irrational number by means of a right triangle.

57. Is a radical always a surd? Illustrate.

58. Is a surd always a radical? Illustrate.

59. Distinguish between a surd and a radical.

60. Which of the numbers $\sqrt{2}$, $\sqrt{4}$, $\sqrt[3]{8}$, $\sqrt{\sqrt[3]{5}}$, $\sqrt{2+\sqrt{2}}$ and $\sqrt{2\pi}$ are surds? Which are radicals?

61. What is the principal square root of 9; the principal cube root of 27; the principal cube root of -8?

62. Name the order of $\sqrt{3}$; of $a^{\frac{1}{4}}$; of $\sqrt[3]{6}$; of $a^{\frac{3}{4}}$; of $\sqrt[3]{a^2}$.

63. Give an example of (*a*) a real number; (*b*) an imaginary number; (*c*) a rational number; (*d*) an irrational number; (*e*) a radical; (*f*) a radicand; (*g*) an index; (*h*) a surd; (*i*) the principal odd root of a positive number; (*k*) the principal even root of a positive number; (*l*) the principal odd root of a negative number.

75. Simplification of radicals. The form of a radical expression may be changed without altering its numeric value. It is often desirable to change the form of a radical so that its numeric value can be computed with the least possible labor.

The simplification of a radical is based on the general identity

$$\sqrt[n]{a^n b} = \sqrt[n]{a^n} \cdot \sqrt[n]{b} = a\sqrt[n]{b}.$$

A radical is in its simplest form when the radicand

I. *Is integral.*

II. *Contains no rational factor raised to a power which is equal to, or greater than, the order of the radical.*

III. *Is not raised to a power, unless the exponent of the power and the index of the root are prime to each other.*

For the meaning of I, II, and III study carefully the following examples.

Examples

Examples of I:

1. $\sqrt{\frac{2}{3}} = \sqrt{\frac{6}{9}} = \sqrt{\frac{1}{9} \cdot 6} = \sqrt{\frac{1}{9}} \cdot \sqrt{6} = \frac{1}{3}\sqrt{6}.$

2. $6\sqrt{\frac{1}{2}} = 6\sqrt{\frac{2}{4}} = 6\sqrt{\frac{1}{4} \cdot 2} = 6\sqrt{\frac{1}{4}} \cdot \sqrt{2} = 6 \cdot \frac{1}{2}\sqrt{2} = 3\sqrt{2}.$

3. $\sqrt{\frac{5}{7\,a}} = \sqrt{\frac{35\,a}{49\,a^2}} = \sqrt{\frac{1}{49\,a^2} \cdot 35\,a} = \frac{1}{7\,a}\sqrt{35\,a}.$

Examples of II:

1. $\sqrt{9\,x^3} = \sqrt{9\,x^2 \cdot x} = \sqrt{(3\,x)^2 \cdot x} = 3\,x\sqrt{x}.$

2. $3\sqrt[3]{54\,m^7} = 3\sqrt[3]{27\,m^6 \cdot 2\,m} = 3\sqrt[3]{(3\,m^2)^3 \cdot 2\,m}$
$= 9\,m^2\sqrt[3]{2\,m}.$

3. $\sqrt{18 - 9\sqrt{3}} = \sqrt{9(2 - \sqrt{3})} = 3\sqrt{2 - \sqrt{3}}.$

Examples of III:

1. $\sqrt[4]{9} = \sqrt[4]{(3)^2} = 3^{\frac{2}{4}} = 3^{\frac{1}{2}} = \sqrt{3}.$

2. $\sqrt[12]{8} = \sqrt[12]{(2)^3} = 2^{\frac{3}{12}} = 2^{\frac{1}{4}} = \sqrt[4]{2}.$

3. $\sqrt[4]{x^2y^6} = \sqrt[4]{(xy^3)^2} = (xy^3)^{\frac{2}{4}} = (xy^3)^{\frac{1}{2}} = y\sqrt{xy}.$

A radical of the second order is simplified by the use of the

RULE. *Separate the radicand into two factors one of which is the greatest perfect square which it contains. Then take the square root of this factor and write it as the coefficient of the radical of which the other factor is the radicand.*

If the original radical has a coefficient other than the number 1, multiply the result obtained above by this coefficient.

A similar rule holds for simplifying radicals involving the cube root and roots of higher orders.

EXERCISES

(Check the values obtained in Exercises 1, 2, 3, 6, 7, 11, and 12 by using the table of square and cube roots on page 411.)

Simplify:

1. $\sqrt{8}$.
2. $\sqrt{20}$.
3. $\sqrt{75}$.
4. $\sqrt{128}$.
5. $\sqrt{108}$.
6. $\sqrt{60}$.
7. $\sqrt{99}$.
8. $\sqrt{125}$.
9. $\sqrt{432}$.
10. $\sqrt{1250}$.
11. $3\sqrt{32}$.
12. $\sqrt[3]{24}$.
13. $8\sqrt[3]{162}$.
14. $\sqrt[4]{405}$.
15. $2\sqrt[5]{729}$.
16. $\sqrt{x^3}$.
17. $\sqrt{ab^3}$.
18. $\sqrt[3]{m^2n^4}$.
19. $2\sqrt[3]{54\,a^5}$.
20. $\sqrt{\frac{1}{5}}$.

Solution. $\sqrt{\frac{1}{5}} = \sqrt{\frac{5}{25}}$
$= \sqrt{\frac{1}{25} \cdot 5}$
$= \frac{1}{5}\sqrt{5}$.

21. $\sqrt{\frac{1}{2}}$.
22. $\sqrt{\frac{7}{3}}$.
23. $\sqrt{\frac{3}{5}}$.
24. $\sqrt[3]{\frac{3}{4}}$.
25. $\sqrt{\frac{1}{x}}$.
26. $\sqrt{\frac{1}{2a}}$.
27. $\sqrt{\frac{2}{x^3}}$.
28. $\sqrt[3]{\frac{1}{m^2}}$.
29. $\sqrt[3]{\frac{1}{mn^3}}$.
30. $\sqrt{\frac{5}{27}}$.
31. $\sqrt[3]{-\frac{4}{9}}$.
32. $\sqrt{(2)^2 - (\frac{3}{5})^2}$.
33. $\sqrt{(6)^2 + (\frac{4}{3})^2}$.
34. $\sqrt{a^2 - \left(\frac{a}{3}\right)^2}$.
35. $\sqrt{\left(\frac{x+4}{4}\right)^2 - x}$.
36. $\sqrt{\left(\frac{e^x - e^{-x}}{2}\right)^2 + 1}$.
37. $\sqrt{9 - 18\sqrt{5}}$.

HINT. $\sqrt{9 - 18\sqrt{5}} = \sqrt{9(1 - 2\sqrt{5})}$.

38. $\sqrt{16 + 8\sqrt{3}}$.

39. $\sqrt{m^3 + m^2\sqrt{2}}$.

40. $\sqrt[3]{250 - 125\sqrt{5}}$.

41. $\sqrt[3]{24 - 16\sqrt{50}}$.

42. $\sqrt{p^2 + 2p^2\sqrt{10}}$.

43. $\sqrt{\dfrac{L^2 + L^2\sqrt{3}}{2}}$.

44. $\sqrt{Q^2 - \dfrac{Q^2}{3}\sqrt{2}}$.

45. $\sqrt[6]{8}$.

Solution. $\sqrt[6]{8} = \sqrt[6]{(2)^3} = 2^{\frac{3}{6}} = 2^{\frac{1}{2}} = \sqrt{2}$.

46. $\sqrt[4]{25}$.

47. $\sqrt[6]{9}$.

48. $\sqrt[4]{x^2y^6}$.

49. $\sqrt[4]{9\,a^{10}}$.

50. $\sqrt[4]{36\,x^2y^4}$.

51. $\sqrt[6]{4\,b^2}$.

52. $\sqrt[4]{\dfrac{r^2}{16}}$.

53. $\sqrt[6]{\dfrac{49\,b^6}{36}}$.

Express entirely under one radical sign:

54. $4\sqrt{3}$.

Solution. $4\sqrt{3} = \sqrt{16} \cdot \sqrt{3} = \sqrt{48}$.

55. $2\sqrt{5}$.

56. $3\sqrt{2}$.

57. $4\sqrt[3]{5}$.

58. $x\sqrt{x}$.

59. $3\,a\sqrt[3]{a^2}$.

60. $9\sqrt[3]{\frac{1}{9}}$.

61. $x\sqrt[3]{x}$.

62. $\dfrac{x}{4}\sqrt[3]{\dfrac{16}{x^2}}$.

63. $e^x\sqrt{e^x - e^{-x}}$.

64. $(b + 2)\sqrt{\dfrac{1}{b^2 - 4}}$.

65. $(3x + 1)\sqrt{\dfrac{5}{9x^2 - 1}}$.

66. $\dfrac{a - 2b}{3}\sqrt[3]{\dfrac{27}{(a - 2b)^2}}$.

67. $\dfrac{x + 5y}{2}\sqrt[5]{\dfrac{32x}{2(x + 5y)^4}}$.

Express in simplest form, with one radical sign:

68. $\sqrt{\sqrt{3}}$.

Solution. $\sqrt{\sqrt{3}} = \sqrt{3^{\frac{1}{2}}} = 3^{\frac{1}{4}} = \sqrt[4]{3}$.

69. $\sqrt{\sqrt{a}}$.
70. $\sqrt[3]{\sqrt{x}}$.
71. $\sqrt{\sqrt{x^3}}$.
72. $\sqrt{\sqrt[3]{y}}$.
73. $\sqrt[3]{\sqrt[3]{z}}$.
74. $\sqrt[3]{\sqrt{m^5}}$.
75. $\sqrt{2\sqrt{2}}$.
76. $\sqrt{5\sqrt{5\sqrt{5}}}$.
77. $\sqrt[4]{\sqrt{27}}$.
78. $2\sqrt[3]{3\sqrt[3]{5}}$.
79. $\sqrt[b]{\sqrt[a]{w^c}}$.
80. $\sqrt[\frac{m}{n}]{x^{\frac{m}{a}}}$.

Find by the formula of Exercise 29, page 118, the areas of the triangles whose sides are:

81. 3, 4, and 5.
82. 13, 14, and 15.
83. 17, 25, and 28.
84. 92, 117, and 205.

76. Addition and subtraction of radicals. *Similar radicals* are radicals of the same order, with radicands which are identical or which can be made so by simplification.

The sum or the difference of similar radicals can be expressed as one term, while the sum or difference of dissimilar radicals can only be indicated.

EXERCISES

Simplify and collect:

1. $\sqrt{3} + \sqrt{27}$.

Solution. $\sqrt{3} + \sqrt{27} = \sqrt{3} + 3\sqrt{3} = 4\sqrt{3}$.

2. $\sqrt{\frac{1}{2}} + 2\sqrt{8}$.
3. $\sqrt{48} + \sqrt{75} - \sqrt{12}$.
4. $2\sqrt{18} - 5\sqrt{50} + 3\sqrt{98}$.
5. $3\sqrt{20} + 12\sqrt{45} + 2\sqrt{125}$.
6. $7\sqrt{108} + 4\sqrt{3} - 10\sqrt{27}$.
7. $2\sqrt{343} + 5\sqrt{175} - 16\sqrt{112}$.

8. $3\sqrt{45} + \sqrt{80} - 2\sqrt{180}$.

9. $2\sqrt{275} + 4\sqrt{44} + 13\sqrt{1331}$.

10. $3\sqrt[3]{2} - \sqrt[3]{16} + 5\sqrt[3]{54}$.

11. $2\sqrt[3]{375} + \sqrt[3]{24} - \sqrt[3]{648}$.

12. $3\sqrt[3]{54} - \sqrt[3]{2000} - \sqrt[3]{250}$.

13. $12\sqrt[3]{6655} + 7\sqrt[3]{135} - 5\sqrt[3]{3645}$.

14. $8\sqrt{\frac{2}{3}} - 2\sqrt{\frac{3}{2}} + 5\sqrt{\frac{1}{6}}$.

15. $2\sqrt{\frac{2}{5}} - 3\sqrt{\frac{1}{10}} + 12\sqrt{\frac{5}{2}}$.

16. $a\sqrt{m^3} + \sqrt{a^2m^3} - 2\,m\sqrt{a^2m}$.

17. $2\,a\sqrt[3]{a^4} - 3\,a\sqrt[6]{a^3} + 9\sqrt{a^3}$.

18. $\sqrt{\frac{2\,a}{b}} - \sqrt{\frac{2\,b}{a}} + \sqrt{\frac{a}{2\,b}}$.

19. $\sqrt{\frac{5\,a^3}{x}} + \sqrt{\frac{a}{5\,x^3}} - \sqrt{\frac{a^5}{5\,x^7}}$.

20. $\sqrt{\frac{12}{7}} + \sqrt{\frac{4}{21}} - \sqrt{\frac{28}{3}}$.

21. $\sqrt[4]{162\,x^9} - \sqrt[4]{32\,x^5} + \sqrt[4]{512\,x} - \sqrt[4]{2\,x}$.

22. $3\,c\sqrt{(a+b)^3} + c\sqrt[6]{(a+b)^3} + \sqrt[4]{(a+b)^2}$.

23. $\sqrt[3]{(x-y)^4} + m\sqrt[6]{x^2 - 2\,xy + y^2} + (x+y)\sqrt[3]{(x-y)}$.

24. $\sqrt{\frac{3}{x}} + \sqrt{\frac{x}{3}} + \sqrt{\frac{x^2+9}{3\,x} + 2} - \sqrt{\frac{x^2+9}{3\,x} - 2}$.

25. $(x-y)\sqrt{\frac{x+y}{x-y}} + \sqrt{9\,x^2 - 9\,y^2} + \frac{x+y}{x-y}\sqrt{\frac{25\,xy^2 - 25\,y^3}{x+y}}$.

77. Multiplication of real radicals. Real radicals of the same order are multiplied as follows:

EXAMPLE 1

Multiply $3\sqrt{v} + 2\sqrt{x} - 5\sqrt{vx}$ by $3\sqrt{vx}$.

Solution.

$$\begin{array}{r} 3\sqrt{v} + 2\sqrt{x} - 5\sqrt{vx} \\ 3\sqrt{vx} \\ \hline 9v\sqrt{x} + 6x\sqrt{v} - 15vx \end{array}$$

Real radicals of different order are multiplied as follows:

EXAMPLE 2

Multiply $\sqrt{m}$ by $\sqrt[3]{v}$.

Solution.

$$\sqrt{m} = m^{\frac{1}{2}} = m^{\frac{3}{6}} = \sqrt[6]{m^3}.$$

$$\sqrt[3]{v} = v^{\frac{1}{3}} = v^{\frac{2}{6}} = \sqrt[6]{v^2}.$$

Then $$\sqrt{m} \cdot \sqrt[3]{v} = \sqrt[6]{m^3} \cdot \sqrt[6]{v^2} = \sqrt[6]{m^3v^2}.$$

The method of multiplying real radicals is stated in the

RULE. *If necessary, reduce the radicals to the same order. Find the products of the coefficients of the radicals for the coefficient of the radical part of the result.*

Multiply together the radicands and write the product under the common radical sign.

Reduce the result to its simplest form.

The preceding rule does not hold for the multiplication of imaginary numbers. This case is discussed in Chapter XI.

EXERCISES

Perform the indicated multiplications and simplify the products:

1. $\sqrt{2} \cdot \sqrt{8}$.
2. $\sqrt{20} \cdot \sqrt{10}$.
3. $\sqrt{\frac{1}{2}} \cdot \sqrt{\frac{2}{3}}$.
4. $\sqrt{\frac{3}{4}} \cdot \sqrt{\frac{3}{2}} \cdot \sqrt{\frac{4}{3}}$.
5. $\sqrt{\frac{1}{3}} \cdot \sqrt{\frac{1}{5}} \cdot \sqrt{\frac{1}{6}}$.
6. $(\sqrt{3} - 3\sqrt{2})\sqrt{3}$.

7. $\left(\sqrt{\frac{2}{3}} - \sqrt{\frac{ab}{2}}\right)\sqrt{\frac{1}{2}}$.

8. $(\sqrt{2} - \sqrt{3})(2\sqrt{2} + 5\sqrt{3})$.

9. $(3\sqrt{5} - 5\sqrt{3})(2\sqrt{5} + 5\sqrt{2})$.

10. $(\sqrt{a} - 5\sqrt{ab})(2\sqrt{a} + 3\sqrt{ab})$.

11. $(\sqrt{2} - \sqrt{5})(\sqrt{2} + \sqrt{5})$.

12. $(4\sqrt{2} - 5\sqrt{10})(4\sqrt{2} + 5\sqrt{10})$.

13. $(\sqrt{3} - \sqrt{7})(\sqrt{7} + \sqrt{3})$.

14. $(3\sqrt{2} - \sqrt{2})(\sqrt{2} + 3\sqrt{2})$.

15. $(4\sqrt{7} - 3\sqrt{10})(4\sqrt{7} + 3\sqrt{10})$.

16. $(2\sqrt{x} - \sqrt{3\,x})^2$.

17. $(\sqrt{a+4})^2$.

18. $(4\sqrt{2\,x - 5})^2$.

19. $(\sqrt{2} - \sqrt{3} + \sqrt{5})(\sqrt{2} + \sqrt{3} - \sqrt{5})$.

20. $(3\sqrt{2} + 5\sqrt{2} - 2\sqrt{6})(2\sqrt{3} + 5\sqrt{2} + 3\sqrt{6})$.

21. $\left(M + \frac{M}{2}\sqrt{5}\right)\left(M - \frac{2\,M}{3}\sqrt{5}\right)$.

22. $\left(\frac{P}{2}\right)^2 + \left(\frac{P}{2} - \frac{P}{2}\sqrt{2}\right)^2$.

23. $R^2 + \left(\frac{R}{2}\sqrt{3} - \frac{R}{2}\right)^2$.

Square:

24. $\sqrt{3} - \sqrt{x+2}$.

25. $\sqrt{a} - \sqrt{5+a}$.

26. $\sqrt{a-2} + \sqrt{2-a}$.

27. $\sqrt{x+2} - \sqrt{2\,x-5}$.

28. $2\sqrt{x+1} - 5\sqrt{5\,x}$.

29. $3\sqrt{x-1} + 5\sqrt{3-x}$.

Perform the indicated multiplications:

30. $(m+\sqrt{n+p})(\sqrt{n+p}-m)$.

31. $(\sqrt{x+y}-\sqrt{x})(\sqrt{x+y}+\sqrt{x})$.

32. $(\sqrt{3y+2z}-2\sqrt{z})(\sqrt{3y+2z}+2\sqrt{z})$.

Express as radicals of the same order:

33. $\sqrt{3}$ and $\sqrt[3]{2}$.

34. $\sqrt[4]{2}$ and $\sqrt[5]{6}$.

35. $\sqrt[3]{3}$ and $\sqrt{8}$.

36. $\sqrt[3]{25}$ and $\sqrt[4]{9}$.

Multiply the following:

37. $\sqrt{2}$, $\sqrt[3]{2}$.

38. $\sqrt{x}$, $\sqrt[3]{a}$.

39. $\sqrt{27}$, $\sqrt[3]{27}$.

40. $\sqrt[4]{x^3}$, $\sqrt{x}$.

41. $\sqrt[4]{10}$, $\sqrt{2}$.

42. $\sqrt[3]{75}$, $\sqrt{\frac{1}{125}}$.

43. $\sqrt[3]{x}$, $\sqrt{y}$.

44. $\sqrt[3]{x^2}$, $\sqrt{x^3}$.

45. $\sqrt{xy}$, $\sqrt[3]{x^2y}$.

46. $\sqrt{\frac{x}{y}}$, $\sqrt[3]{\frac{y}{x}}$.

47. $\sqrt[4]{3m^2}$, $\sqrt[3]{2m^5}$.

48. $\sqrt{a-x}$, $\sqrt[3]{x+a}$.

78. Division of radicals. Direct division of radicals coefficient by coefficient and radicand by radicand is often possible.

Thus $12\sqrt{2} \div 3\sqrt{5} = 4\sqrt{\frac{2}{5}} = \frac{4}{5}\sqrt{10}$,

and $5\sqrt{st} \div 4\sqrt{t} = \frac{5}{4}\sqrt{s}$.

Direct division of radicals when the divisor is a radical expression with more than one term is usually very difficult. In such cases a rationalizing factor of the denominator is used as described in the following section. We then carry out the operation of division indirectly by resorting to multiplication.

79. Rationalizing factor. One radical expression is a **rationalizing factor** for another if the product of the two is rational.

A rationalizing factor for $\sqrt{5}$ is $\sqrt{5}$, since $\sqrt{5} \cdot \sqrt{5} = 5$. For $\sqrt[3]{2}$ a rationalizing factor is $\sqrt[3]{4}$, since $\sqrt[3]{2} \cdot \sqrt[3]{4} = \sqrt[3]{8} = 2$. Similarly, $\sqrt{5} - \sqrt{3}$ is a rationalizing factor of $\sqrt{5} + \sqrt{3}$, as their product, $(\sqrt{5} - \sqrt{3})(\sqrt{5} + \sqrt{3})$, is equal to $5 - 3$, or 2.

In like manner $(2\sqrt{5} + 3\sqrt{2})(2\sqrt{5} - 3\sqrt{2}) = 20 - 18 = 2$.

Two important radical expressions are $\sqrt{a} + \sqrt{b}$ and $\sqrt{a} - \sqrt{b}$. Two such binomials are called **conjugate** radicals, and either is a rationalizing factor for the other.

Rationalizing factors are used in division of radicals as follows:

EXAMPLES

1. $\sqrt{3} \div \sqrt{10} = \dfrac{\sqrt{3}\sqrt{10}}{\sqrt{10}\sqrt{10}} = \dfrac{\sqrt{30}}{10}$.

2. $(3\sqrt{5} + 4\sqrt{7}) \div 4\sqrt{15} = \dfrac{(3\sqrt{5} + 4\sqrt{7}) \cdot \sqrt{15}}{4\sqrt{15} \cdot \sqrt{15}}$

$= \dfrac{15\sqrt{3} + 4\sqrt{105}}{60}$.

3. $(\sqrt{2} - \sqrt{3}) \div (\sqrt{3} + \sqrt{5}) = \dfrac{(\sqrt{2} - \sqrt{3})(\sqrt{3} - \sqrt{5})}{(\sqrt{3} + \sqrt{5})(\sqrt{3} - \sqrt{5})}$

$= \dfrac{\sqrt{6} - 3 - \sqrt{10} + \sqrt{15}}{3 - 5}$

$= -\frac{1}{2}(\sqrt{6} - \sqrt{10} + \sqrt{15} - 3)$.

Therefore when direct division of radicals is impossible, use the

RULE. *Write the dividend over the divisor in the form of a fraction. Then multiply the numerator and denominator of the*

fraction by a rationalizing factor for the denominator and simplify the resulting fraction.

This rule applies in all cases, while the rule for direct division fails when dividing a real radical by an imaginary number, as will be seen in section 88.

EXERCISES

Find a simple rationalizing factor for:

1. $\sqrt{2}$.
2. $2\sqrt{5}$.
3. $\sqrt{11}$.
4. $4\sqrt{3}$.
5. $\sqrt[3]{4}$.
6. $\sqrt[4]{27}$.
7. $\sqrt{2}-\sqrt{5}$.
8. $\sqrt{6}-1$.
9. $2\sqrt{5}-3\sqrt{7}$.
10. $\sqrt{2a}-\sqrt{3x}$.
11. $\sqrt{2-a}-\sqrt{3}$.
12. $\sqrt{2-a}-3\sqrt{2a}$.
13. $\sqrt{x-y}+\sqrt{x+y}$.

Perform the indicated division:

14. $\sqrt{27}\div\sqrt{3}$.
15. $3\sqrt{12}\div\sqrt{2}$.
16. $5\sqrt{6}\div\sqrt{3}$.
17. $\sqrt{15}\div\sqrt{30}$.
18. $\sqrt{m^3x}\div\sqrt{mx^3}$.
19. $\sqrt{5ab}\div\sqrt{3ab^2}$.
20. $(\sqrt{12}-\sqrt{8})\div\sqrt{2}$.
21. $(4\sqrt{5}-10\sqrt{75})\div(3\sqrt{5})$.
22. $(\sqrt{x^2y}+\sqrt{xy^2})\div\sqrt{xy}$.
23. $12\div6\sqrt{3}$.
24. $15\div3\sqrt{5}$.
25. $75\div5\sqrt{27}$
26. $\sqrt{5}\div\sqrt[3]{3}$.

Solution. $\dfrac{\sqrt{5}}{\sqrt[3]{3}}=\dfrac{\sqrt{5}\cdot\sqrt[3]{9}}{\sqrt[3]{3}\cdot\sqrt[3]{9}}=\dfrac{\sqrt[6]{125\cdot81}}{3}=\dfrac{\sqrt[6]{10125}}{3}$

27. $\sqrt{3}\div\sqrt[3]{4}$.
28. $\sqrt{x}\div\sqrt[4]{3}$.
29. $\sqrt[4]{a}\div\sqrt{5}$.
30. $\sqrt[3]{x}\div\sqrt[4]{a}$.

31. $\sqrt[3]{21} \div \sqrt[5]{m^3}$.

32. $\sqrt{50} \div \sqrt[3]{2}$.

33. $\sqrt[3]{\frac{1}{9}} \div \sqrt{\frac{1}{3}}$.

34. $m \div 2c\sqrt{p}$.

35. $\sqrt{2} \div (\sqrt{2} - 3)$.

36. $\sqrt{7} \div (\sqrt{7} + \sqrt{8})$.

37. $(3\sqrt{2} + \sqrt{7}) \div (\sqrt{2} - \sqrt{7})$.

Change to respectively equivalent fractions having rational denominators:

38. $\dfrac{\sqrt{2}}{\sqrt{3} - \sqrt{7}}$.

39. $\dfrac{\sqrt{10}}{\sqrt{2} + \sqrt{5}}$.

40. $\dfrac{\sqrt{2} + \sqrt{3}}{\sqrt{3} - \sqrt{2}}$.

41. $\dfrac{6\sqrt{2} + 5\sqrt{7}}{5\sqrt{2} - 6\sqrt{7}}$.

42. $\dfrac{\sqrt{x} + \sqrt{y}}{\sqrt{x} - \sqrt{y}}$.

43. $\dfrac{\sqrt{x} + 5\sqrt{z}}{\sqrt{x} + 2\sqrt{z}}$.

44. $\dfrac{\sqrt{a+b} + \sqrt{c}}{\sqrt{a+b} - \sqrt{c}}$.

45. $\dfrac{\sqrt{m+b}}{6 - \sqrt{m+b}}$.

46. $\dfrac{9}{\sqrt{3} - \sqrt{3}}$.

47. By the use of the table, page 411, calculate the expressions in Exercises 41 and 46 in the form given and in the required form with rational denominators.

Rationalize the denominators of:

48. $\dfrac{5}{\sqrt[4]{2} + 1}$

49. $\dfrac{7}{\sqrt[4]{3} - 2}$

Perform the indicated division:

50. $(\sqrt{11} + \sqrt{6}) \div (\sqrt{11} - \sqrt{6})$.

51. $(a - \sqrt{b}) \div (a - 2\sqrt{b})$.

52. $(\sqrt{a+x} - \sqrt{z}) \div (\sqrt{a+x} + \sqrt{z})$.

53. $(\sqrt{10} - \sqrt{7} - \sqrt{2}) \div (\sqrt{7} + \sqrt{2})$.

54. Is there any difference in meaning between the direction before Ex. 38 and that before Ex. 50?

55. Does $3 + \sqrt{13}$ satisfy $x^2 - 6x - 4 = 0$?

56. Does $\dfrac{\sqrt{2} - 2}{3}$ satisfy $3x^2 - 4x + 2 = 0$?

57. Does $\frac{1}{7}(1 \pm \sqrt{22})$ satisfy $7x^2 - 2x - 3 = 0$?

80. Square root of surd expressions. The square of a binomial is usually a trinomial. However, the result of squaring a binomial of the form $\sqrt{a} + \sqrt{b}$ is a binomial if a and b are rational numbers. Thus

$$(\sqrt{5} - \sqrt{2})^2 = 5 - 2\sqrt{10} + 2 = 7 - 2\sqrt{10}.$$

In $7 - 2\sqrt{10}$, 7 is the sum of 5 and 2, and 10 is the product of 5 and 2. These relations and the fact that the coefficient of the radical $\sqrt{10}$ is 2 enable us to find the square root of many expressions of the form $a \pm \sqrt{b}$ by writing in the form of $x \pm 2\sqrt{xy} + y$ and then extracting the square root of the trinomial square as follows.

EXAMPLE

Extract the square roots of $8 - \sqrt{60}$.

Solution. $8 - \sqrt{60} = 8 - 2\sqrt{15}$.

We must now find two numbers whose sum is 8 and whose product is 15. These are 3 and 5.

Then $8 - 2\sqrt{15} = 5 - 2\sqrt{15} + 3 = (\sqrt{5} - \sqrt{3})^2$.

Hence the square roots of $8 - \sqrt{60}$ are $\pm(\sqrt{5} - \sqrt{3})$.

EXERCISES

Find the ~~positive~~ square roots in Exercises 1–13.

1. $7 - 2\sqrt{6}$.
2. $7 - 2\sqrt{12}$.
3. $6 - \sqrt{32}$.
4. $17 + 2\sqrt{16}$.
5. $12 - 8\sqrt{2}$.
6. $21 + 14\sqrt{2}$.
7. $13 + \sqrt{160}$.
8. $22 + \sqrt{228}$.
9. $14x + \sqrt{96x^2}$.
10. $22m + 2m\sqrt{57}$.
11. $4x - 2\sqrt{\frac{35}{9}x^2}$.
12. $2p - 2\sqrt{p^2 - 1}$.
13. $6 + 2\sqrt{9 - a^2}$.
14. $\sqrt{9+4\sqrt{5}} = \sqrt{?} + \sqrt{?}$.
15. $\sqrt{13 + 2\sqrt{30}} = ?$
16. $\sqrt{20 + 10\sqrt{3}} = ?$
17. $\sqrt{x + \sqrt{x^2 - 4b^2}} = ?$
18. $\sqrt{a^3 + a^2 + 3a + 2a\sqrt{3a^2 + 3a}} = ?$

NOTE. In the writings of one of the later Hindu mathematicians (about A.D. 1150) we find a method of extracting the square root of surds which is practically the same as that given in the text. In fact, the formula for the operation is given, apart from the modern symbols, as follows: $\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}$. The study of expressions of the type $\sqrt{\sqrt{a} \pm \sqrt{b}}$ had been carried to a most remarkable degree of accuracy by the Greek, Euclid. His researches on this subject, if original with him, place him among the keenest mathematicians of all time, but his work and all of his results are expressed in geometric language which is very far removed from the algebraic symbolism of to-day.

PROBLEMS

(Obtain answers in simplest radical form and correct to the nearest second decimal place, where the results are numeric and involve decimals.)

1. The side of an equilateral triangle is 5 inches. Find the altitude and the area.

2. The side of an equilateral triangle is b. Find the altitude and the area.

3. The altitude of an equilateral triangle is 36. Find one side and the area.

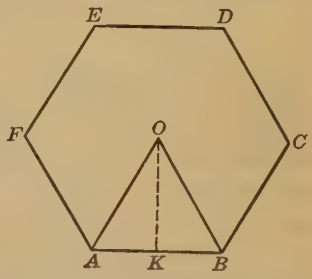
4. The area of an equilateral triangle is A. What is the altitude and one side?

5. A triangle has sides 4 inches, 8 inches, and 10 inches. Find the area of the triangle. Find the altitude on the side of 10 inches.

HINT. Let the altitude on the side 10 be x, and the two parts into which the altitude divides side 10 be y and $10 - y$; then set up the equations involving x and y, and solve.

6. Find the altitude on the shortest side of the triangle whose sides are 9 inches, 5 inches, and 12 inches.

Fact from Geometry. A regular hexagon may be divided into six equal equilateral triangles by lines from its center to the vertices.



In the adjacent regular hexagon, $AB = BC = CD$, etc. O is the center and OK perpendicular to AB is the **apothem** of the hexagon.

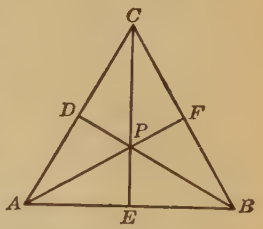
7. Find the apothem and the area of a regular hexagon (a) whose side is 20, (b) whose side is m.

8. Find the side and the area of a regular hexagon (a) whose apothem is 25, (b) whose apothem is a.

Facts from Geometry. The volume of a pyramid or cone is $\frac{ab}{3}$, where a is the altitude and b is the area of the base.

The altitudes of an equilateral triangle intersect at a point which divides each altitude into two parts whose ratio is 2 to 1.

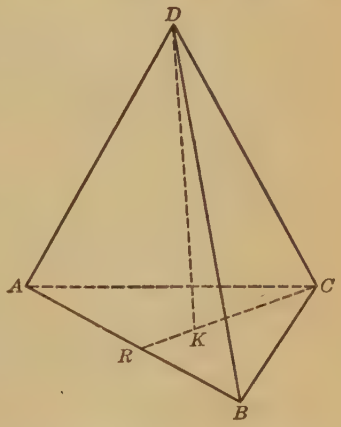
9. The sides of a regular pyramid with a square base are equilateral triangles of sides 32 feet. Find the altitude and the volume of the pyramid.



10. The side of an equilateral triangle is 40 inches. Find the two parts into which each altitude is divided by the other altitudes. What is the area of each of the six smaller triangles thus formed?

A **regular tetrahedron** is a pyramid whose four faces are equal equilateral triangles.

The altitude of a regular tetrahedron (*DK* in the adjacent figure) meets the base at the point where the altitudes of the base intersect.



11. *ABCD* is a regular tetrahedron. If each edge is 36, find *CR*, *CK*, and the altitude *DK*.

12. Find the altitude and volume of a regular tetrahedron each of whose edges is 25 inches.

13. Show that the altitude and volume of a regular tetrahedron whose edge is e are

$$\frac{e}{3}\sqrt{6} \text{ and } \frac{e^3}{12}\sqrt{2} \text{ respectively.}$$

MISCELLANEOUS EXERCISES

Reduce to equivalent fractions having rational denominators:

1. $\dfrac{2\sqrt{3}}{\sqrt{2}}$.

2. $\dfrac{3}{7\sqrt{5}}$.

3. $\dfrac{4}{\sqrt{2}-\sqrt{5}}$.

4. $\dfrac{6}{6^{\frac{1}{2}}-5^{\frac{1}{2}}}$.

5. $\dfrac{3^{\frac{1}{2}}+7^{\frac{1}{2}}}{2\cdot 3^{\frac{1}{2}}-3\cdot 7^{\frac{1}{2}}}$.

6. $\dfrac{3\sqrt{7}+2\sqrt{3}}{3\sqrt{7}-2\sqrt{3}}$.

7. $\dfrac{x\sqrt{y}-a\sqrt{z}}{x\sqrt{y}+a\sqrt{z}}$.

8. $\dfrac{d-\sqrt{d^2-4}}{d+\sqrt{d^2-4}}$.

9. $\dfrac{a+b}{a^{\frac{1}{2}}+b^{\frac{1}{2}}}$.

10. $\dfrac{x-x^{\frac{1}{2}}y^{\frac{1}{2}}+y}{x^{\frac{1}{2}}-y^{\frac{1}{2}}}$.

Find the positive square roots of the following:

11. $25+2\sqrt{66}$.

12. $49-14\sqrt{6}$.

13. $16+8\sqrt{3}$.

14. $25-10\sqrt{6}$.

15. Show that

$$\frac{2+\sqrt{7}}{3} \text{ is a root of } 3x^2-4x-1=0.$$

16. Show that

$$\frac{2\pm 5\sqrt{3}}{4} \text{ are roots of } 16x^2-16x=71.$$

17. Show that

$2a\pm 3\sqrt{b}$ are roots of $x^2-4ax+4a^2=9b$.

18. Show that $\dfrac{-2m-\sqrt{24mn+m^2n}}{3n}$ is a root of the equation $9n^2x^2+12mnx+4m^2=m^2n+24mn$.

Simplify, using the table, page 411, in securing results correct to two decimal places:

19. $3\sqrt{50} - 5\sqrt{8} + \sqrt{2} - \frac{1}{7}\sqrt{98}$.

20. $4\sqrt{\frac{2}{3}} - 2\sqrt{\frac{1}{6}} + 21\sqrt{\frac{3}{2}} + \sqrt{150}$.

21. $\sqrt{10} \div \sqrt{20}$.

22. $\sqrt{7} \div 5\sqrt{3}$.

23. $(8\sqrt{6} - \sqrt{7}) \div (\sqrt{7} + \sqrt{2})$.

Simplify:

24. $\sqrt{15} + 5\sqrt{\frac{1}{15}} - \sqrt{\frac{3}{5}} + 3\sqrt{\frac{5}{3}}$.

25. $y \cdot \frac{1}{5}(x^2 - y^2)^{-\frac{1}{2}}(-10\,y) - (x^2 - y^2)^{\frac{1}{2}}$.

26. $\left(\dfrac{m^5 \cdot n \cdot m^{-1} - n \cdot 5\,m^4}{(m^4)^3}\right) \div \dfrac{n}{m^3}$.

27. $\dfrac{3\,x^2(x^4) - x^4(4\,x^2 - 15)}{(x^2)^5} \div \dfrac{x^2 - 15}{x^5}$.

28. $\dfrac{\sqrt{m}}{\sqrt{m} + \sqrt{n}} - \dfrac{\sqrt{n}}{\sqrt{m} - \sqrt{n}}$.

29. $\dfrac{(i^a + i^{-a})(i^a + i^{-a}) - (i^a - i^{-a})(i^a - i^{-a})}{(i^a + i^{-a})}$.

30. $\dfrac{\dfrac{(m^3 - 1)m^{(b-1)}\,b + m^b \cdot 3\,m}{(m^3 - 1)^2}}{\dfrac{m^b}{m^3 - 1}}$.

31. $\dfrac{\dfrac{y^3(ay^{a-2}) - (y^a + 1)2\,y}{(y^3)^2}}{\dfrac{y^a + 1}{y^3}}$.

32. $\dfrac{\dfrac{m^{3a}bm^{b-1} - m^b \cdot 3\,am^{2a-1}}{(m^{2a})^3}}{\dfrac{m^b}{m^{3a}}}$.

33. $\dfrac{\dfrac{2\,z^{-3}(-3\,z^{-1}) - (z^3+2)(-6\,z^{-6})}{(z^{-4})^3}}{\dfrac{z^{-3}+2}{z^{-4}}}$.

34. $\dfrac{\dfrac{e^{-ab}(ae^{ab}) - (e^{ab}+2)(-ae^{-ab})}{(e^{-ab})^2}}{\dfrac{3\,e^{ab}+2}{e^{-ab}}}$.

35. $\dfrac{\dfrac{\frac{1}{2}(\sqrt[5]{x}+1)^2\,x^{-\frac{2}{5}} - \sqrt[5]{x}\,(\frac{1}{2}\,x^{-\frac{2}{5}})}{(\sqrt[5]{x}+1)^3}}{\dfrac{\sqrt{mx}}{(\sqrt[5]{x}+1)^2}}$

CHAPTER IX

QUADRATIC EQUATIONS

81. Solution by completing the square. An equation of the form $ax^2 + bx + c = 0$, where a, b, and c denote numbers or known literal expressions, is called a **quadratic equation.** Any such equation can be solved by the method of completing the square. This method gets its name from the fact that in the course of the solution there is added to each member of the equation a number such that one member becomes a perfect square.

ORAL EXERCISES

What terms should be added in order to make the following expressions perfect trinomial squares?

1. $a^2 + 2a +$
2. $m^2 - 2m +$
3. $x^2 + 4x +$
4. $x^2 + 6x +$
5. $b^2 + 8b +$
6. $x^2 + x +$
7. $a^2 - a +$
8. $b^2 + 3b +$
9. $x^2 - \frac{2}{3}x +$
10. $x^2 + \frac{4}{5}x +$
11. $m^2 + \frac{a}{2}m +$
12. $p^2 - dp +$

13. How is the term which is added in the preceding exercises obtained from the coefficients of the two given terms?

Example

Solve $7x^2 - 2x - 5 = 0$ and check the result.

Solution. $7x^2 - 2x - 5 = 0.$ (1)

Transposing, $7x^2 - 2x = 5.$

Dividing by the coefficient of x^2,

$$x^2 - \tfrac{2}{7}x = \tfrac{5}{7}.$$

Adding $(-\tfrac{1}{7})^2$ to each member

$$x^2 - \tfrac{2}{7}x + \tfrac{1}{49} = \tfrac{5}{7} + \tfrac{1}{49} = \tfrac{36}{49}$$

or $$(x - \tfrac{1}{7})^2 = \tfrac{36}{49}.$$

Extracting the square root of each member,

$$x - \tfrac{1}{7} = \pm \tfrac{6}{7}.$$

Whence $$x = 1 \text{ and } -\tfrac{5}{7}.$$

Check. Substituting 1 for x in (1),

$$7 \cdot (1)^2 - 2 \cdot 1 - 5 = 0.$$
$$7 - 2 - 5 = 0.$$
$$0 = 0.$$

Substituting $-\tfrac{5}{7}$ for x in (1),

$$7 \cdot (-\tfrac{5}{7})^2 - 2(-\tfrac{5}{7}) - 5 = 0.$$
$$\tfrac{25}{7} + \tfrac{10}{7} - 5 = 0.$$
$$0 = 0.$$

The method of solving a quadratic equation in x illustrated in the preceding example is stated in the

Rule. *Transpose terms so that the terms containing* x *are in the first member and those which do not contain* x *are in the second.*

Divide both members of the equation by the coefficient of x^2 *unless that coefficient is* $+1$.

Then add to both members the square of one half the coefficient of x *in the equation just obtained, thus making the first member a perfect trinomial square.*

Rewrite the equation, expressing the first member as the square of a binomial and the second member in its simplest form.

Extract the square root of both members of the equation, and write the sign ± before the square root of the second member, thus obtaining two linear equations.

Solve the equation in which the second member is taken with the sign +, and then solve the equation in which the second member is taken with the sign −. The results are the roots of the quadratic equation.

CHECK. *Substitute each result separately in place of* x *in the original equation. If the resulting equations are not obvious identities, simplify until they become so.*

EXERCISES

Solve by completing the square and check real results as directed by the teacher:

1. $x^2 + 6x - 7 = 0.$
2. $x^2 - 4x - 6 = 0.$
3. $x^2 - 28 = 3x.$
4. $x^2 - 2x = 11.$
5. $2x^2 - 2x - 5 = 0.$
6. $2x^2 - 10x = 15.$
7. $6x^2 + 6 + 13x = 0.$
8. $6x^2 + 13x + 2 = 0.$
9. $3x^2 - 14x = 5.$
10. $8a^2 + 22a - 21 = 0.$
11. $2m^2 - 5m + 2 = 0.$
12. $5p - 2 = 3p^2.$
13. $3x^2 - 10x + 5 = 9x - 10 - 3x^2.$
14. $x^2 = \frac{x + 15}{2}.$
15. $(2m + 1)^2 - (m - 1)^2 = -2.$
16. $x^2 = \frac{14 - 19x}{40}.$

In Exercises 17–26 obtain results to the nearest hundredth:

$$17.\ x^2 - 4x + 2 = 0.$$

HINTS. By applying the rule we get

$$x = 2 + \sqrt{2}.$$
$$x = 2 - \sqrt{2}.$$

From the table $\sqrt{2} = 1.4142.$

This gives us $x = 3.41$ and $.59.$

$18.\ x^2 - 2x - 2 = 0.$

$19.\ p^2 - 12p + 29 = 0.$

$20.\ r^2 - 4r - 1 = 0.$

$21.\ m^2 + 8m + 10 = 0.$

$22.\ 10m^2 - 29m + 10 = 0.$

$23.\ 2x^2 - 2 - x\sqrt{2} = 0.$

$24.\ x^2 + 5x\sqrt{3} - 5 = 0.$

$25.\ 3x^2 + 2x\sqrt{2} - 2 = 0.$

$26.\ \dfrac{x^2}{12} - \dfrac{x}{3} = \dfrac{3}{4}.$

$$27.\ x^4 - 5x^2 + 6 = 0.$$

NOTE. This is not a quadratic equation, but many equations of this form can be solved by the methods applicable to quadratics.

Solution.

$$x^4 - 5x^2 + 6 = 0.$$
$$x^4 - 5x^2 = -6.$$
$$x^4 - 5x^2 + \tfrac{25}{4} = \tfrac{25}{4} - \tfrac{24}{4} = \tfrac{1}{4}.$$
$$(x^2 - \tfrac{5}{2})^2 = \tfrac{1}{4}.$$
$$x^2 - \tfrac{5}{2} = \pm\tfrac{1}{2}.$$
$$x^2 = 2 \text{ and } 3.$$

Whence $x = \pm\sqrt{2}$ and $\pm\sqrt{3}.$

Check as usual.

NOTE. It should be particularly observed that the equation of Exercise 27 has four roots instead of two. In general an equation has a number of roots equal to its degree. Thus the equations in Exercises 30 and 32 have six and eight roots respectively, although some of them are imaginary. Consequently the student at present cannot expect to find them all.

28. $x^4 - 6x^2 + 5 = 0$.

29. $x^4 - 4x^2 - 45 = 0$.

30. $x^6 + x^3 - 12 = 0$.

31. $x^6 - 2x^3 = 15$.

32. $m^8 - 21m^4 + 20 = 0$.

33. $2x^4 - 7x^2 + 6 = 0$.

34. $3b^6 - 5b^3 + 2 = 0$.

35. $x^4 - 6x^2 + 8 = 0$.

36. $y^6 - 8y^3 + 12 = 0$.

37. $2b^4 - 5b^2\sqrt{2} + 6 = 0$.

38. $4(x^2 - x)^2 - 19(x^2 - x) = -22$.

Solution.

Let $x^2 - x = y$.

Substituting y for $x^2 - x$, we get

$$4y^2 - 19y = -22.$$

Solving, $y = 2$ and $\frac{11}{4}$.

Then $x^2 - x = 2$.

Whence $x = -1$ and 2.

Also $x^2 - x = \frac{11}{4}$.

Whence $x = \frac{1}{2} \pm \sqrt{3}$.

In Exercises 39–42 do not expand but solve as in Exercise 38:

39. $(x + 2)^2 + 3(x + 2) - 4 = 0$.

40. $(y^2 - 3y)^2 - 13(y^2 - 3y) = -30$.

41. $2(m^2 - 2m)^2 - 13(m^2 - 2m) + 15 = 0$.

42. $(2x + 6)^2 + 20(2x + 6) + 91 = 0$.

43. $ax^2 + bx + c = 0$.

Solution. $ax^2 + bx + c = 0$.

$$x^2 + \frac{b}{a}x = -\frac{c}{a}.$$

$$x^2 + \frac{b}{a}x + \left(\frac{b}{2a}\right)^2 = \frac{b^2}{4a^2} - \frac{c}{a} = \frac{b^2 - 4ac}{4a^2}.$$

$$x + \frac{b}{2a} = \frac{\pm\sqrt{b^2 - 4ac}}{2a}$$

$$x = -\frac{b}{2a} \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

44. $2x^2 + ax - 6a^2 = 0.$

45. $cz^2 - cz = 3 - 3z.$

46. $m^2y + my^2 + mn + ny = 0.$

47. $cw^2 + 3aw = c - 3a.$

48. $abx^2 + x(a^2 - b^2) = ab.$

49. $x^2\sqrt{pq} + pqx(\sqrt{q} - \sqrt{p}) = p^2q^2.$

50. $2x^4 + a^2x^2 = 6a^4.$

51. $6m^2 + bmx - 2b^2x^2 = 0.$

52. $(3pz + 4m)^2 = (2pz - m)^2.$

53. $\frac{3}{4}x^4 - \frac{1}{2}ax^2 = 4a^2.$

54. $2b = ax\sqrt{b} + a^2x^2.$

55. $2k^2 + 3kx\sqrt{3} + 3x^2 = 0.$

56. $(2x + c)^2 + 2x + c = 2.$

57. $\frac{a^2}{y^2} = \frac{a - 3}{y - 3}.$

58. $\frac{2y + 5}{2a + 5} = \frac{y^2}{a^2}.$

59. $\frac{x^2}{3x - 1} = \frac{n^2}{3n - 1}.$

60. $\frac{4x - 7}{4a - 7} = \frac{x^2}{a^2}.$

61. $\frac{n^2 + 2n + 4}{x^2 + 2x + 4} = \frac{n^2}{x^2}.$

62. $\frac{3a + 2}{a^2} = \frac{3x + 2}{x^2}.$

63. $\frac{x^2 - x + 1}{a^2 - a + 1} = \frac{x^2}{a^2}.$

64. $\frac{x^2 + 3}{a^2 + 3} = \frac{x}{a}.$

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82. Solution by formula. In ~~Exercise 43, above~~, the general quadratic $ax^2 + bx + c = 0$ has been solved and the roots found to be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (F)$$

The expression (F) is a general result and may be used as a formula to solve any quadratic equation which is in the standard form $ax^2 + bx + c = 0$, where a, b, and c may represent numbers, single letters, binomials, or any other form of algebraic expression not involving x.

If the numbers a, b, and c are such that the expression $b^2 - 4ac$ is negative, the formula contains the square root of a negative number, which is a kind of number not yet fully considered in this text. In the exercises that follow it will be assumed that only such numerical values of the literal coefficients are involved as will not make the radicand $b^2 - 4ac$ negative. A discussion of the case here ruled out will be found in Chapter XI.

EXERCISES

Solve for x by formula and check results as directed by the teacher:

1. $2x^2 + 5x = 1$.

Solution. Writing in standard form,

$$2x^2 + 5x - 1 = 0.$$

Comparing with $ax^2 + bx + c = 0$, we see that 2 corresponds to a, 5 to b, and -1 to c. Substituting these values in the formula (F) gives

$$x = \frac{-5 \pm \sqrt{25 - 4 \cdot 2 \cdot (-1)}}{4},$$

$$= \frac{-5 \pm \sqrt{33}}{4}.$$

Check as usual.

Solve by formula:

2. $x^2 - 2x - 15 = 0$.
3. $x^2 + x - 1 = 0$.
4. $x^2 + 7x + 1 = 0$.
5. $3x^2 - 4x - 2 = 0$.
6. $4(x + 2) = x^2$.
7. $x = x^2 - 1$.
8. $x^2 - 5x - 5 = 0$.
9. $4x^2 - 8x - 1 = 0$.
10. $4x^2 - 12x + 7 = 0$.
11. $3x^2 + 6x + 2 = 0$.
12. $9x^2 - 6x - 4 = 0$.
13. $2x^2 - 4x + 1 = 0$.
14. $5x^2 - 16x + 11 = 0$.
15. $9x^2 - 42x + 38 = 0$.

16. $2k^2x^2 - kx - 6 = 0$.

Solution. $2k^2x^2 - kx - 6 = 0$.

Here $a = 2k^2$, $b = -k$, and $c = -6$.

Substituting these values in formula (F),

$$x = \frac{-(-k) \pm \sqrt{(-k)^2 - 4 \cdot 2k^2(-6)}}{2 \cdot 2k^2}$$

$$= \frac{k \pm \sqrt{k^2 + 48k^2}}{4k^2} = \frac{k \pm 7k}{4k^2} = \frac{2}{k} \text{ and } -\frac{3}{2k}.$$

Check as usual.

17. $x^2 - 3bx - 4b^2 = 0$.
18. $6x^2 + 7px - 18p^2 = 0$.
19. $10x^2 - 3ax - a^2 = 0$.
20. $6p^2x^2 - 7px + 2 = 0$.
21. $3b^2x^2 - 2abx = a^2$.
22. $q(x^2 + p^2) = pq(q^2 + 1)$.
23. $x + 2\sqrt{a} = \frac{3a}{x}$.
24. $\sqrt{2}x^2 - 3ax + a^2 = 0$.
25. $3abx^2 + 2ab = (3b^2 + 2a^2)x$.

26. $x^2 + \frac{bx}{c} + \frac{1}{c^2} = 0.$

27. $16\,x^2 + rx = s.$

28. $a(a - 2\,x) = b^2 - x^2.$

29. $2\,x^2 + ac = (2\,c + a)x.$

30. $x^2 + px + 2 - px^2 - 3\,x = 0.$

Solution. $(1 - p)x^2 + (p - 3)x + 2 = 0.$

Here $a = 1 - p$, $b = p - 3$, and $c = 2$.

Hence
$$x = \frac{3 - p \pm \sqrt{(p - 3)^2 - 8(1 - p)}}{2(1 - p)}$$
$$= \frac{3 - p \pm \sqrt{p^2 + 2\,p + 1}}{2(1 - p)}.$$
$$x = \frac{3 - p \pm (p + 1)}{2 - 2\,p}$$
$$= \frac{2}{1 - p} \text{ and } 1.$$

Check as usual.

31. $x^2 + a^2 + x - a = 2\,ax.$

32. $(x - 1)^2 = a(x^2 - x).$

33. $x^2 + x - 2 + k(x^2 + 2\,x) = 0.$

34. $a^2x^2 - 2\,ax + 1 = ax^2 - x.$

35. $x^2 + kx - 2\,k^2 = x + 2\,k.$

36. $b^2x^2 + 2 = x^2 + 3\,bx + x.$

37. $x^2 + xy + x + 3\,y - 6 = 0.$

38. $x^2 - 6\,x + 3\,xy - 12\,y + 8 = 0.$

39. $x^2 + 5\,x - 2\,xy - 4\,y + 6 = 0.$

40. $2\,x^2 + x + 2\,xy + 3\,y = 3.$

41. $x^2 - 3x - y^2 - 3y = 0.$

42. $x^2 - axy + (a + b)x - a^2y + ab = 0.$

43. $x^2 - 5xy + x + 6y^2 - 2 - y = 0.$

83. Comparison of the various methods. Three methods have been given for the solution of the quadratic equation:

(*a*) Solution by factoring (p. 53).
(*b*) Solution by completing the square (p. 143).
(*c*) Solution by formula (p. 149).

In practice, the first and the last of these methods are most convenient. A graphical method for the solution of a quadratic equation is presented on pages 183–185.

When an equation with integral coefficients can be solved by factoring, as explained in Chapter III, the roots are always rational numbers.

For example, $10x^2 - 11x - 6 = 0$ factors into $(2x - 3)(5x + 2) = 0$. Hence the roots are $x = \frac{3}{2}$ and $-\frac{2}{5}$.

An inspection of the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

shows that, if a, b, and c are integers, one always obtains roots involving radicals unless the expression under the radical sign, $b^2 - 4ac$, is a perfect square. In this case, however, the values of the roots are rational numbers.

For example, in the equation $10x^2 - 11x - 6 = 0$ the value of the expression $b^2 - 4ac$ is $(-11)^2 - 4 \cdot 10(-6) = 361$, which is a perfect square. Hence the roots of $10x^2 - 11x - 6 = 0$ are rational numbers, since the radical term in the roots can be expressed as a rational number.

Hence to determine whether a quadratic equation of the form $ax^2 + bx + c = 0$ has rational roots, we have the

RULE. *Compute the value of* $b^2 - 4ac$ *for the equation. If the result is the square of an integer, the left member of the equation can be factored and the roots are rational.*

ORAL EXERCISES

Determine which of the following have rational roots:

1. $x^2 + 3x + 2 = 0$.
2. $x^2 - 4x + 10 = 0$.
3. $x^2 - 4x - 4 = 0$.
4. $2x^2 + 5x + 2 = 0$.
5. $2x^2 + 5x + 3 = 0$.
6. $x^2 - 2x - 1 = 0$.
7. $4x^2 - x - 3 = 0$.
8. $2x^2 - x + 1 = 0$.
9. $2x^2 - 6x - 9 = 0$.
10. $3x^2 - 5x - 12 = 0$.
11. $9x^2 - 9x - 3 = 0$.
12. $10x^2 + 6x + 1 = 0$.

REVIEW EXERCISES

Solve the following by the method best adapted to each:

1. $x^2 - 8x + 12 = 0$.
2. $6x^2 + x - 35 = 0$.
3. $3x^2 - 2x - 5 = 0$.
4. $5x^2 + 7x + 1 = 0$.
5. $x^2 - 2.5x = 1.86$.
6. $3x^2 - 5\sqrt{2}x + 1 = 0$.
7. $4x^3 = x$.
8. $x + 2 = 6x^2$.
9. $x^3 - 2x^2 - x + 2 = 0$.
10. $x^2 - x - 1 = 0$.
11. $\frac{3x^2}{5} = 15(x - 2)(x + 2)$.
12. $x^2 - 5.5x + 7.36 = 0$.
13. $(x + 2)^2 + 3(x + 2) - 4 = 0$.
14. $3(x - 3)^2 - 11(x - 3) - 4 = 0$.

15. $(x^2 - 1)^2 - 4(x^2 - 1) - 5 = 0.$

16. $.09\,x^2 - .21\,x + .1 = 0.$

17. $x^2 - x + .24 = 0.$

18. $3\,x^2 - 12.3\,x + 7.8 = 0.$

19. $\dfrac{x-1}{x} = \dfrac{x-1}{6}.$

20. $\dfrac{x-2}{x-3} + \dfrac{x+2}{x+3} = \dfrac{1}{4}.$

21. $\dfrac{x+1}{x-1} - \dfrac{x-1}{x+1} = \dfrac{5}{6}.$

22. $\dfrac{3\,x-2}{x-1} - \dfrac{2\,x+1}{x+1} = 0.$

23. $\dfrac{x-7}{2\,x-10} = \dfrac{x+5}{3\,x+21}.$

24. $\dfrac{3}{x+2} + \dfrac{x-1}{x+5} = \dfrac{5\,x+15}{6\,x+18}.$

25. $\dfrac{2.1}{x+.3} + \dfrac{1-1.4\,x}{.5} - 3.88 = 0.$

26. $\dfrac{x}{2\,x-.3} = \dfrac{3.4}{x}.$

27. $\dfrac{x+2}{x+1} - \dfrac{2\,x+1}{x+2} = \dfrac{x-6}{3\,x-4}.$

28. $x^2 - 4\,x + 4 = a^2.$

29. $x^2 - 8\,ax + 16\,a^2 - 9\,b^2 = 0.$

30. $2\,x^2 + 6\,x - 4\,a^2 = x^2 - 9.$

31. $10\,x^2 + 11\,x + .028 = 0.$

32. $\dfrac{x^2 - 2.8}{x} = \dfrac{3.1}{2}.$

33. $\dfrac{(x-2)^3 - (x+1)^3}{x} = \dfrac{2\,x-31}{2}.$

34. $\dfrac{x^3 - 6}{x} = 3 - 2\,x.$

35. $5(2x + 1)(x - 5) = (4x - 3)(5x + 29)$.

36. $\frac{2}{x+3} + \frac{4}{3} = \frac{3}{x-7}$.

37. $2(3 - 4x)(2x + 1) = (6x - 1)(2x - 11)$.

38. $\frac{2x^2}{3} - x\sqrt{2} = 1$.

39. $\frac{x+\sqrt{2}}{x-\sqrt{3}} + \frac{x+\sqrt{3}}{x-\sqrt{2}} + \frac{5}{\sqrt{6}} = 0$.

40. $\frac{x^2}{3x-5} = \frac{m^2}{3m-5}$.

41. $\frac{x^2}{n^2} = \frac{2x+a}{2n+a}$.

42. $\frac{x^2}{c^2} = \frac{x^2+x+1}{c^2+c+1}$.

43. $\frac{x^2}{m^2} = \frac{ax+b}{am+b}$.

44. $(3x + 4)^2 - (2x + 3)^2 = 0$.

45. $(3x + 4)^2 - (2x + 3)^2 = 1$.

46. $\frac{3}{x-1} - \frac{x-5}{x^2+6x-7} = \frac{2}{5}$.

47. $39abx = 10a^2 + 14b^2x^2$.

48. $9x^2 - 2k^2 + 2km + 3kx + 3mx = 0$.

49. $\frac{3x^2+x-2}{2x^2-x+3} = \frac{6x^2+2x-2}{4x^2-2x+2}$.

50. $59x^2 = 15x^4 + 52$.

51. $7.3x^2 - 11.1x - 6.3 = 0$.

52. $.51x^2 + .73x + .16 = 0$.

53. $\frac{2x-3}{x} - \frac{2x}{2x-1} = \frac{10x-39}{5(2x+3)}$.

54. $4 + \frac{a}{a-2x} = \frac{a+2x}{a}$.

55. $\frac{x+1}{(x+3)(x-1)} - \frac{2x+1}{(x+3)(2-x)} = \frac{6x}{(1-x)(x-2)}$.

56. $(4x+7)(x+2) - (x+3)(2x+5) = 0$.

57. $a^2x^2 + 1 = b^2x^2 + 2ax$.

58. $x + \sqrt{a} = \frac{1}{\sqrt{a}} - \frac{1}{x}$.

59. $\frac{.3x-2}{.1x-1} + \frac{.5x+5}{.4x-1} = 8\frac{1}{2}$.

60. $2kx + 2x - x^2 = k^2 + 2k$.

61. $x^2 - 3a = ax - 3x$.

62. $\frac{x}{\sqrt{3}} = \frac{2\sqrt{5}}{x} + 3$.

63. $\frac{a}{x+a} + \frac{b}{x+b} - \frac{2c}{x+c} = 0$.

64. $\frac{a+x}{b+x} + \frac{b+x}{a+x} = \frac{5}{2}$.

65. $\frac{ax+b}{bx+a} = \frac{mx-n}{nx-m}$.

66. $\frac{ax^2+bx+c}{bx^2+mx+n} = \frac{c}{n}$.

67. $\frac{2x+a}{ax+2} = \frac{a+x}{2+x}$.

PROBLEMS

1. Separate 56 into two parts such that one part shall be the square of the other.

2. Separate 45 into two parts such that one part shall be twice the square of the other.

3. Find two consecutive even numbers whose product is 168.

4. Find two integers whose difference is 4 and whose product is 117.

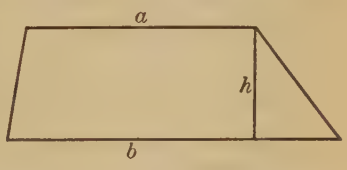
5. The product of two consecutive numbers is one greater than the sum of the integers next smaller and next greater than these numbers. Find the two numbers.

6. One side of a rectangle is 6 feet longer than the other. The area of the rectangle is 91 square feet. Find the sides.

7. The altitude of a triangle is 4 feet more than its base. The area is 16 square feet. Find the altitude and the base.

8. One base of a trapezoid is twice as long as the altitude, while the other base is 17 feet longer than the altitude. The area of the trapezoid is 45 square feet. Find the lengths of the bases.

HINT. The area of a trapezoid is given by the formula $A = \frac{h(a+b)}{2}$, where h is the altitude and a and b are the bases.

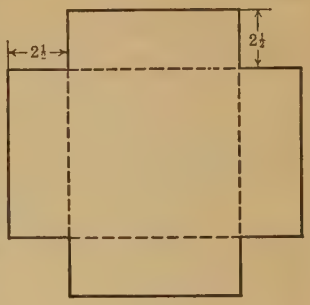


9. A polygon of n sides always has $\frac{1}{2}\,n(n-3)$ diagonals. How many sides has a polygon with 135 diagonals?

10. The side of one square is 8 inches greater than that of a second square. The sum of the areas of the two squares is 424 square inches. Find the side of each square.

11. The radius of one circle is 2 inches greater than the radius of a second circle. The sum of their areas is $106\frac{6}{7}$ square inches. Find the radius of each circle.

12. A $2\frac{1}{2}$ inch square is cut from each corner of a square piece of tin. The sides are then turned up to form an open box of volume 80 cubic inches. What is the side of the original square?



13. A tinsmith wishes to make a tin box 4 inches deep which will hold 180 cubic inches, by cutting squares from the corners of a square of tin and folding up the sides. How large a square of tin will he require?

14. What positive value of x will make the product of $x - 3$ and $x + 3$ greater by 10 than their difference?

15. What values of x will make the product of $3x - 2$ and $x + 3$ equal to $7x + 2$?

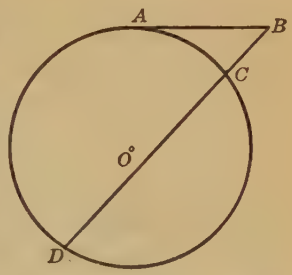
16. The length of a room is 4 feet greater than its width. A rug placed in the middle of the room leaves a margin 2 feet wide on each side. If the entire margin is just equal to the area of the rug, find the dimensions of the room.

17. A lawn is 36 by 24 yards. How wide a strip must be cut around it with a lawn mower in order to leave just $\frac{5}{8}$ of it uncut?

18. A farmer has fencing enough to go around a lot whose length is 3 rods more than twice its width. If he uses the same amount of fencing to inclose a square lot, the area inclosed will be increased by 36 square rods. How many rods of fencing has he?

19. The side of a cabin is six feet high, and is to be built of boards of uniform width. If the boards had each been one inch wider it would have required one less board to cover the side of the cabin. How wide were the boards?

20. If AB in the accompanying figure is a tangent to the circle, and BD is any secant, then $(AB)^2 = BC \cdot BD$. If $AB = 6$ and $CD = 9$, find BC.



21. The distance from the horizon to the top of a cliff is 20 miles. How high is the cliff? (Earth's radius = 4000 miles.)

22. Ninety-six children were to be taken to a picnic in automobiles, each containing the same number of children. Four cars did not appear, and four more children rode in each car than was planned. How many cars were ordered?

23. The denominator of a fraction is 4 greater than the numerator. If 5 be subtracted from both numerator and denominator, the fraction is $\frac{1}{3}$ its original value. What is the fraction?

24. The sum of \$2000 is invested, and at the end of the first year the year's interest plus \$900 is added to the investment. At the end of the second year the investment amounts to \$3100. What is the rate of interest?

25. The distance in feet, s, through which a body falls from rest in t seconds, neglecting the resistance of the air, is given by the formula $s = 16\,t^2$. A bomb is dropped from an aëroplane and strikes the ground 12 seconds later. How high is the plane?

26. If a projectile is thrown vertically into the air with a velocity of 200 feet per second, its distance above the ground after t seconds is $s = 200\,t - 16\,t^2$ feet, neglecting air-resistance. After how many seconds will it be 600 feet above the ground? After how many seconds will the projectile return to the ground? Explain the zero root.

CHAPTER X

IRRATIONAL EQUATIONS

84. Definitions and discussion. An **irrational** equation in one unknown is an equation in which the unknown occurs under a radical, or is affected by a fractional exponent.

Thus $\sqrt{x} - 2x + 1 = 0$, and $x^{\frac{1}{4}} - x^{\frac{1}{2}} + 1 = 0$, and $y - (2y)^{\frac{1}{2}} - 4 = 0$ are irrational equations.

One difficulty involved in the solution of such equations arises from the fact that sometimes results are obtained which do not satisfy the given equation and hence are not roots of that equation. A result of this kind is called **extraneous.**

EXAMPLE

(*a*) Solve $\sqrt{x-3} - 5 = 0$. (*b*) Solve $-\sqrt{x-3} - 5 = 0$.

Solution. Transposing,

	$\sqrt{x-3} = 5.$	(1)	$-\sqrt{x-3} = 5.$	(1)
Squaring,	$x - 3 = 25.$	(2)	$x - 3 = 25.$	(2)
Solving,	$x = 28.$		$x = 28.$	
Check.	$\sqrt{28-3} - 5 = 0.$		$-\sqrt{28-3} - 5 = 0.$	
	$\sqrt{25} - 5 = 0.$		$-\sqrt{25} - 5 = 0.$	
	$5 - 5 = 0,$		$-5 - 5 = 0,$	
	which is true		which is not true.	

It appears from a study of these solutions that statements (1) differ only in the signs preceding their left members. Consequently this distinction disappears after squaring, and equations (2) are identical. Since the remainder of the work in both (*a*) and (*b*) consists in the solution of (2), the result obtained is really the root of this equation. Whether the root obtained satisfies both (*a*) and (*b*), or only one of them, can be determined only by substitution. In this case it appears that (*a*) is an equation and that (*b*) is not, but is merely a false statement in the form of an equation.

In any case, all of the roots of the original equation are sure to be among the results found, provided no factor containing the unknown has been divided out. But **no result should be called a root unless it satisfies the original equation.** This means that all results must be checked.

In irrational equations, as in all the work up to the present, it is understood that unless a radical or an expression affected by a fractional exponent is preceded by the double sign $\pm$ it has only the one value, just like any other number symbol.

Thus $\sqrt{81}$ means $+9$, and not -9.

Also $9^{\frac{1}{2}}$ means $+3$, while $-9^{\frac{1}{2}}$ means $-\sqrt{9}$, or -3.

and $x^{\frac{1}{2}}$ means $+\sqrt{x}$ and not $-\sqrt{x}$.

If this fact is kept in mind, it is clear from an inspection of (*b*), above, that it could have no root, since the sum of two negative numbers could not possibly be zero.

The method of solving equations in which an unknown occurs under a radical is stated in the

RULE. ***Transpose the terms so that one radical expression (the least simple one if there are two or more) is the only term in one member of the equation.***

Next raise both members of the resulting equation to the same power as the index of this radical.

Combine like terms in each member, and, if radical expressions still remain, repeat the two preceding operations until an equation is obtained which is free from radicals; then solve this equation.

CHECK. *Substitute in the original equation the values found and reduce the resulting numeric equation to its simplest form by extracting roots, but not by raising both members of the equation to any power.*

Finally, ~~reject all extraneous roots.~~ explain all roots

EXERCISES

Solve the following for real roots and check results, ~~rejecting all extraneous roots~~:

1. $\sqrt{x+2} = 3.$
2. $\sqrt{5x+11} = 6.$
3. $\sqrt{3x+7} = 10.$
4. $3\sqrt{2x+3} - 2 = 7.$
5. $\sqrt[3]{4x+4} = 2.$
6. $2\sqrt[3]{3x+1} + 5 = 7.$
7. $x + \sqrt{x+8} + 2 = 0.$

Solution.

$$\sqrt{x+8} = -(x+2).$$
$$x + 8 = x^2 + 4x + 4.$$
$$x^2 + 3x - 4 = 0;$$
$$x = -4, \text{ or } 1.$$

Check.

Substituting -4 for x,

$$-4 + \sqrt{4} + 2 = -2 + 2 = 0,$$

which is true.

Substituting 1 for x,

$$1 + \sqrt{9} + 2 = 0$$
$$6 = 0,$$

which is not true.

Hence 1 is extraneous.

8. $x - \sqrt{x} - 2 = 0$.

9. $\sqrt{x-3} = \sqrt{4x-5}$.

10. $2(x-4)^{\frac{1}{2}} = (2x+3)^{\frac{1}{2}}$.

11. $x\sqrt{6} = \sqrt{x+2}$.

12. $\sqrt{2}(x+1)^{\frac{1}{2}} = \sqrt{3}(5x+2)^{\frac{1}{2}}$.

13. $\sqrt{x^2+4x-5} = \sqrt{2-2x}$.

14. $x = 11 - 3\sqrt{x+7}$.

15. $5 - \sqrt{2x+5} = \sqrt{2x}$.

16. $\sqrt{x+1} = \sqrt{x^2+3x+2}$.

17. $\sqrt{x+1} = \sqrt[4]{x+1}$.

18. $(3x-4)^{\frac{1}{2}} + (4x+3)^{\frac{1}{2}} = 0$.

19. $\sqrt{x-2} = \sqrt[4]{x^2-6x+1}$.

20. $\sqrt{2x+5} = 1 - \sqrt{x+2}$.

Solution. $\sqrt{2x+5} = 1 - \sqrt{x+2}$.

Squaring, $2x + 5 = 1 - 2\sqrt{x+2} + x + 2$.

Transposing and collecting,

$$x + 2 = -2\sqrt{x+2}.$$

Squaring,

$$x^2 + 4x + 4 = 4x + 8,$$

or

$$x = \pm 2.$$

Check. Substituting $+2$ for x in the original equation,

$$\sqrt{9} = 1 - \sqrt{4} \text{ or } 3 = -1,$$

which is not true.

Substituting -2 for x,

$$\sqrt{1} = 1 - \sqrt{0}, \text{ or } 1 = 1.$$

Hence -2 is the only root.

21. $\sqrt{x-2} = \sqrt{x} + 2$.

22. $\sqrt{2x+3} + 3 = 3\sqrt{x+1}$.

23. $\sqrt{x-1}+\sqrt{x}=\sqrt{2x-1}$.

24. $\sqrt{x-1}+\sqrt{3x+1}-2=0$.

25. $\sqrt{3(x+2)}+\sqrt{x+4}=\sqrt{7x+1}$.

26. $x-4=\dfrac{(x+4)^{\frac{1}{2}}}{3}$.

27. $\sqrt{x+3}+\sqrt{x}=\sqrt{3}$.

28. $1+\sqrt{x+2}=\sqrt{x}$.

29. $\sqrt{x+2}-\sqrt{3x+4}=\sqrt{2x+2}$.

30. $x^{\frac{4}{3}}-10x^{\frac{2}{3}}+9=0$.

NOTE. The equation here given is not a quadratic equation, but it is of the general type $ax^{2n}+bx^{n}+c=0$. Here x occurs in but two terms, and its exponent in one term is twice that in the other term. Many equations in this form can be solved by completing the square (compare Exercise 27, p. 146).

Solution. $x^{\frac{4}{3}}-10x^{\frac{2}{3}}+9=0.$

$(x^{\frac{2}{3}}-9)(x^{\frac{2}{3}}-1)=0.$

Hence $x^{\frac{2}{3}}=9$ and 1.

Thus $x^{\frac{1}{3}}=\pm 3$ and $x=\pm 27$

or $x^{\frac{1}{3}}=\pm 1$ and $x=\pm 1$.

Check. Substituting ± 27 for x in the original equation,

$(\pm 27)^{\frac{4}{3}}-10(\pm 27)^{\frac{2}{3}}+9=0,$

or $81-10\cdot 9+9=0.$

Substituting ± 1 for x in the original equation,

$(\pm 1)^{\frac{4}{3}}-10(\pm 1)^{\frac{2}{3}}+9=0$

or $1-10+9=0.$

31. $x^{\frac{2}{3}} - 7\,x^{\frac{1}{3}} - 8 = 0.$

32. $x^5 + 4\,x^{\frac{5}{2}} - 5 = 0.$

33. $x = x^{\frac{1}{2}} + 2.$

34. $x^{-2} + 5\,x^{-1} + 6 = 0.$

35. $\frac{3}{x} - 10 = x^{-\frac{1}{2}}.$

36. $6\,x^3 + 7\,x^{\frac{3}{2}} + 1 = 0.$

37. $8\,x^3 - 7\,x^{\frac{3}{2}} - 1 = 0.$

38. $8\,x^3 + 7\,x^{\frac{3}{2}} - 1 = 0.$

39. $x^{\frac{1}{3}} + x^{\frac{1}{6}} - 2 = 0.$

40. $\sqrt[4]{x} = \sqrt{x} - 2.$

41. $\sqrt{3 - 2\,x} = x + 30.$

42. $6\,x^2 - \frac{165}{x^2} = 3.$

43. $3\,x - 11\,x^{\frac{1}{2}} - 20 = 0.$

44. $\sqrt{2\,x - 1} + \sqrt{x + 4} = 6.$

45. $\sqrt{3\,x + 1} - \sqrt{2\,x - 1} = 1.$

46. $\sqrt{3\,x - 2} + \sqrt{2\,x + 5} = 5.$

47. $\sqrt{3\,x + 4} - \sqrt{2\,x - 4} = 2.$

48. $\frac{3\sqrt{x + 4}}{\sqrt{3\,x} - \sqrt{6}} = \sqrt{6}.$

49. $\frac{\sqrt{x} - 3}{\sqrt{x}} - \frac{5 - \sqrt{x}}{4} = 0.$

50. $\frac{\sqrt{x} - 1}{\sqrt{x} + 1} = \frac{\sqrt{2}}{\sqrt{3}}.$

51. $\frac{2 - \sqrt{x}}{\sqrt{2\,x}} = \frac{3\sqrt{x} - 2}{\sqrt{2}}.$

52. $\frac{x^{\frac{1}{2}} - 5}{2 - 2\,x^{\frac{1}{2}}} = \frac{3\,x^{\frac{1}{2}}}{7}.$

53. $\frac{2\,a^{\frac{1}{2}}}{(2\,x - a)^{\frac{1}{2}}} = \frac{(2\,x + 4\,a)^{\frac{1}{2}}}{3\,a^{\frac{1}{2}}}.$

54. $\frac{(x - 4)^{\frac{1}{2}}}{(2\,x + 3)^{\frac{1}{2}}} + \frac{(2\,x + 3)^{\frac{1}{2}}}{(x - 4)^{\frac{1}{2}}} = 2.$

CHAPTER XI

IMAGINARIES

85. Introduction. As we have progressed in the study of algebra, nearly every forward step has involved the use of a more complicated and refined type of number. In arithmetic the positive integer and the positive fraction were sufficient for all requirements. In the beginning of algebra the negative integer and fraction were introduced. The solution of quadratic equations such as, $x^2 - 2 = 0$, forced upon us the irrational number, and led to the study of methods of operating with radicals, and to the solution of radical equations. Each new kind of number has presented itself as the root of an equation that we were supposed to solve, and with the introduction of each new number the power and generality of our algebraic method was increased.

Up to the present the square root of a negative number has been avoided, or dismissed with the remark that it is an imaginary. It is true that we cannot imagine any length that the number $\sqrt{-2}$ could measure. The positive numbers are all that we need for measurement. Similarly the person with very simple mathematical needs might remark, that since positive integers are adequate for counting, the fractions and irrational numbers are unnecessary.

Just as we have defined and used the operations of addition, subtraction, multiplication, and division with

negative numbers (Chapter I) and with irrational numbers (Chapter VIII), so now we will define the meaning of these operations on the so-called imaginaries. The introduction of these numbers enables us to solve completely the quadratic equation for all cases. They are frequently used in many branches of applied mathematics, especially in the theory of electricity.

The equation $x^2 + 1 = 0$, or $x^2 = -1$, states that x is a number whose square is -1. By defining a new number, $\sqrt{-1}$, as one whose square is -1, we obtain one root for the equation $x^2 + 1 = 0$.

Similarly, $\sqrt{-5}$ is a number whose square is -5. And, in general, $\sqrt{-n}$ is a number whose square is $-n$. Obviously, $\sqrt{-5}$ means something very different from $\sqrt{5}$, and $\sqrt{-n}$ from $\sqrt{n}$.

The positive numbers are all multiples of the unit $+1$, and the negative numbers are all multiples of the unit -1. Similarly, pure imaginary numbers are real multiples of the imaginary unit $\sqrt{-1}$, as $2\sqrt{-1}$, $-5\sqrt{-1}$, and $b\sqrt{-1}$.

Furthermore, $\sqrt{-4} = \sqrt{4 \cdot (-1)} = 2\sqrt{-1}$; $\sqrt{-a^2} = \sqrt{a^2(-1)} = a\sqrt{-1}$; and $\sqrt{-5} = \sqrt{5} \cdot \sqrt{-1}$.

The imaginary unit $\sqrt{-1}$ is often denoted by the letter i; that is, $3\sqrt{-1} = 3i$.

If a real number be united to a pure imaginary by a plus sign or a minus sign, the expression thus obtained is called a **complex number.**

Thus $-2 + \sqrt{-1}$ and $3 - 2\sqrt{-4}$ are complex numbers. The general form of a complex number is $a + bi$, in which a and b may be any real numbers.

NOTE. Up to the time of Gauss (1777–1855) complex numbers were not clearly understood and were usually thought of as absurd.

The situation reminds one of the time when negative numbers were similarly regarded, and the veil was removed from both in about the same way. It was found that negative numbers really had a significance — that they could be used in problems that involve debt, opposite directions, and many other everyday relations. The interpretation of imaginary numbers is not quite so obvious, and is not considered in this text. But as soon as it was seen that an interpretation was possible the ice was broken, and it needed only the insight and authority of a man like Gauss to give complex numbers their proper place in mathematics.

ORAL EXERCISES

Express as multiples of $\sqrt{-1}$ or i:

1. $\sqrt{-9}$.
2. $\sqrt{-36}$.
3. $\sqrt{-x^2}$.
4. $3\sqrt{-2}$.
5. $2\sqrt{-5}$.
6. $5\sqrt{-b}$.
7. $\sqrt{6}\cdot\sqrt{-6}$.
8. $\sqrt{3}\cdot\sqrt{-2}$.
9. $\sqrt{7}\cdot\sqrt{-28}$.
10. $x\sqrt{-y}$.
11. $\sqrt{-a^2-4a-4}$.
12. $\sqrt{-x^2-12x-36}$.

86. Addition and subtraction of imaginaries. The fundamental operations of addition and subtraction are performed with imaginary and complex numbers as they are performed with rational numbers and ordinary radicals of similar form.

Thus $3\sqrt{-1}+2\sqrt{-1}=5\sqrt{-1}$,

and $7\sqrt{-1}-\sqrt{-16}=7\sqrt{-1}-4\sqrt{-1}=3\sqrt{-1}$,

and $3(-1)^{\frac{1}{2}}+5(-1)^{\frac{1}{2}}-4\sqrt{-1}=4i$.

Also $(2+3\sqrt{-1})+(5-2\sqrt{-1})=7+\sqrt{-1}$.

Similarly $(x+yi)+(m+ni)=m+x+(n+y)i$.

EXERCISES

Simplify and express results as multiples of i:

1. $4\sqrt{-1}+3\sqrt{-1}$.
2. $3\sqrt{-1}+\sqrt{-4}$.
3. $\sqrt{-16}-\sqrt{-9}$.
4. $\sqrt{-25}+\sqrt{-36}$.
5. $\sqrt{-9}+\sqrt{-4}$.
6. $(-27)^{\frac{1}{2}}+(-18)^{\frac{1}{2}}$.
7. $\sqrt{-8}+\sqrt{-32}$.
8. $2\sqrt{-36\,a^2}-5\sqrt{-49\,a^2}$.
9. $3+2\sqrt{-4}+5-6\sqrt{-1}$.
10. $3\sqrt{-x^2}+2x-5\sqrt{-x^2}$.
11. $(2m+3ni)+(m-2ni)$.
12. $16-9i+5-3\sqrt{-25}$.
13. $7-4\sqrt{-16\,a^2}-5\sqrt{-36\,a^2}+2\sqrt{-4\,a^2}+6$.
14. $30-2(-1)^{\frac{1}{2}}+7(-16)^{\frac{1}{2}}+5$.
15. $3\sqrt{-3}+2\sqrt{-8}-3\sqrt{-32}+4\sqrt{-48}$.
16. $(9-3\sqrt{-25})+(8+2\sqrt{-9})$.
17. $\sqrt{-16\,a^6}+3\,a^3\sqrt{-4}+2\sqrt{-12}+5\sqrt{-27}$.
18. $(2a+3bi)-(5a-6bi)$.

87. Multiplication of imaginaries. By the definition of the square root of a number, the square of $(-n)^{\frac{1}{2}}$ is $-n$.

Therefore $(\sqrt{-1})^2=-1$,

$$(\sqrt{-1})^3=(\sqrt{-1})^2\sqrt{-1}=-\sqrt{-1},$$

and $$(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=(-1)(-1)=1.$$

To multiply $\sqrt{-3}$ by $\sqrt{-5}$, we first write

$$\sqrt{-3}=\sqrt{3}\cdot\sqrt{-1}$$

and $$\sqrt{-5}=\sqrt{5}\cdot\sqrt{-1}.$$

Then $\sqrt{-3}\cdot\sqrt{-5}=(\sqrt{3}\cdot\sqrt{-1})(\sqrt{5}\cdot\sqrt{-1})$
$=\sqrt{15}\cdot\sqrt{-1}\cdot\sqrt{-1}=\sqrt{15}(-1)$
$=-\sqrt{15}.$

Similarly
$(4\sqrt{-2})(-\sqrt{3}\sqrt{-7})=(4\sqrt{2}\cdot\sqrt{-1})(-\sqrt{3}\sqrt{7}\sqrt{-1})$
$=-4\sqrt{42}(-1)=+4\sqrt{42}.$

In general, if $\sqrt{-a}$ and $\sqrt{-b}$ are two imaginaries whose product is desired, they should first be written in the form $\sqrt{a}\cdot\sqrt{-1}$ and $\sqrt{b}\cdot\sqrt{-1}$ or $\sqrt{a}\,i$ and $\sqrt{b}\,i$ and the multiplication should not be performed until this has been done. This method will prevent many errors.

In this connection it must be clearly understood that one rule followed in the multiplication of real radicals (see page 129) does not apply to imaginary numbers.

In the case of ordinary radicals we have

$$\sqrt{2}\cdot\sqrt{3}=\sqrt{2\cdot 3}=\sqrt{6}.$$

But the product of two imaginaries like $\sqrt{-2}\cdot\sqrt{-3}$ does not equal $\sqrt{(-2)(-3)}$, for this equals $\sqrt{6}$. We have seen above that $\sqrt{-2}\cdot\sqrt{-3}=\sqrt{2}\cdot\sqrt{3}(\sqrt{-1})^2=-\sqrt{6}$.

In multiplying two complex numbers, write each expression in the form $a\pm bi$ and proceed as in the following

EXAMPLE

Multiply $3+\sqrt{-2}$ by $5-\sqrt{-5}$.

Solution. $3+\sqrt{-2}=3+\sqrt{2}\sqrt{-1}$ (1)

and $5-\sqrt{-5}=5-\sqrt{5}\sqrt{-1}$. (2)

Multiplying (1) by (2),

$$15+5\sqrt{2}\sqrt{-1}-3\sqrt{5}\sqrt{-1}-\sqrt{10}(-1).$$

Rewriting

$$15+\sqrt{10}+(5\sqrt{2}-3\sqrt{5})\sqrt{-1}.$$

EXERCISES

Perform the following indicated multiplications and simplify results:

1. $(-1)^4$.
2. $(-1)^5$.
3. $(-1)^6$.
4. $(-1)^7$.
5. $(\sqrt{-1})^6$.
6. $(\sqrt{-1})^5$.
7. $(\sqrt{-2})^7$.
8. $(\sqrt{-3})^3$.
9. $(3\sqrt{-7})^5$.
10. $\sqrt{-9}\cdot\sqrt{-16}$.
11. $\sqrt{-25}\,(-\sqrt{-4})$.
12. $\sqrt{-5}\,(-\sqrt{-3})$.
13. $\sqrt{-9}\cdot\sqrt{-15}$.
14. $3\sqrt{-5}\cdot 5\sqrt{-3}$.
15. $\sqrt{-x}\cdot\sqrt{-y}$.
16. $5\sqrt{-10}(-3\sqrt{-7})$.
17. $\sqrt{a+b}\cdot\sqrt{-a-b}$.
18. $(2+\sqrt{-1})(2-\sqrt{-1})$.
19. $(4+\sqrt{-2})(4-\sqrt{-2})$.
20. $(\sqrt{5}+\sqrt{-3})(\sqrt{5}-\sqrt{-3})$.
21. $(2+\sqrt{-4})(4-\sqrt{-2})$.
22. $(7+\sqrt{-3})(5+\sqrt{-6})$.
23. $(2-6i\sqrt{3})(2+3i\sqrt{3})$.
24. $(5+\sqrt{-1})(3-\sqrt{-1})$.
25. $(6+2i)(3-2i\sqrt{3})$.
26. $(m+ni)(p+qi)$.
27. $(x+bi)^2$.
28. $(m+xi)(m-xi)$.
29. $(-\frac{1}{2}+\frac{1}{2}\sqrt{-2})^2$.
30. $(-\frac{1}{2}-\frac{1}{2}\sqrt{2})^2$.
31. $(-3+3\sqrt{-5})^3$.
32. $(-3-3\sqrt{-5})^3$.
33. $(m-ni)^3$.
34. $(x+yi)^2-(x-yi)^2$.
35. $(2+2\sqrt{-5})^3-(2-2\sqrt{-5})^3$.
36. $(m+i\sqrt{1-x^2})(m-i\sqrt{1-x^2})$.

37. Determine whether (*a*) the sum, and (*b*) the product, of the numbers $3+4\sqrt{-2}$ and $3-4\sqrt{-2}$ are real.

38. Determine whether (*a*) the sum, and (*b*) the product, of the numbers $1+\sqrt{-5}$ and $1-\sqrt{-5}$ are real.

88. Division of imaginaries. One complex number is called the *conjugate* of another if their product and their sum are real. Thus $a + bi$ and $a - bi$ are conjugates. Conjugate complex numbers are used in division of imaginary expressions as conjugate radicals are used in division of real radicals.

In case either the numerator or the denominator of a fraction is imaginary or complex, the division may be performed as in the following

EXAMPLES

1. $\sqrt{-10} \div \sqrt{5}$.

Solution. $$\frac{\sqrt{-10}}{\sqrt{5}} = \frac{\sqrt{5}\ \sqrt{-10}}{(\sqrt{5})^2} = \frac{5\sqrt{-2}}{5}$$
$$= \sqrt{2}\,i.$$

2. $\sqrt{27} \div \sqrt{-3}$.

Solution. $$\frac{\sqrt{27}}{\sqrt{-3}} = \frac{\sqrt{27} \cdot \sqrt{-3}}{(\sqrt{-3})^2} = \frac{9\,i}{-3}$$
$$= -3\,i.$$

3. $\sqrt{-15} \div \sqrt{-5}$.

Solution. $$\frac{\sqrt{-15}}{\sqrt{-5}} = \frac{\sqrt{15}\,i}{\sqrt{5}\,i} = \sqrt{3}.$$

4. $2 \div (3+\sqrt{-5})$.

Solution. $$\frac{2}{3+\sqrt{-5}} = \frac{2}{3+\sqrt{5}\,i} \cdot \frac{3-\sqrt{5}\,i}{3-\sqrt{5}\,i}$$
$$= \frac{6-2\sqrt{5}\,i}{9+5}$$
$$= \frac{3-\sqrt{5}\,i}{7}.$$

The method of the above examples is stated in the

RULE. *Write the dividend over the divisor in the form of a fraction.*

Then multiply both numerator and denominator of this fraction by the simplest expression which will make the new denominator real and rational.

Reduce the result to its simplest form.

EXERCISES

Perform the indicated operations:

1. $\sqrt{-6} \div \sqrt{2}$.
2. $\sqrt{15} \div \sqrt{-3}$.
3. $6\sqrt{3} \div 3\sqrt{-1}$.
4. $\sqrt{-8} \div \sqrt{-2}$.
5. $2 \div \sqrt{-2}$.
6. $1 \div \sqrt{-7}$.
7. $\sqrt{27} \div \sqrt{-3}$.
8. $(-25)^{\frac{1}{2}} \div (-81)^{\frac{1}{2}}$.
9. $\sqrt{ab} \div \sqrt{-a}$.
10. $\sqrt{-x} \div \sqrt{-y}$.
11. $(-10\,am)^{\frac{1}{2}} \div (-3\,m)^{\frac{1}{2}}$.
12. $[(-b^4)^{\frac{1}{2}} - (-b^8)^{\frac{1}{2}}] \div (-b)^{\frac{1}{2}}$.
13. $5 \div (1 - \sqrt{-1})$.
14. $3 \div (2 + \sqrt{-3})$.
15. $3\sqrt{-2} \div (\sqrt{-1} - 3)$.
16. $4\sqrt{-3} \div (2\sqrt{-5} + 3)$.
17. $(-2 + 3\sqrt{-2}) \div (-2 - 3\sqrt{-2})$.
18. $(1 - 3\,i) \div (5 + 7\,i)$.
19. $a \div (a + bi)$.
20. $(x + iy) \div (m + in)$.
21. $(2 + 3\,i)(i - 1) \div (2 - 5\,i)(i + 1)$.
22. Is $\frac{3}{2}(1 + \sqrt{-3})$ a cube root of -27?

23. Does $x^2 - 4x + 16 = 0$ if $x = 2 + 2\sqrt{-3}$?

24. Does $-1 + 2\sqrt{-1}$ satisfy $x^2 + 2x + 5 = 0$?

25. Does $x = -\frac{2}{3}\sqrt{-2}, y = \frac{1}{3}\sqrt{-2}$ satisfy the system

$$9x^2 - 18xy - 27y^2 = -10,$$
$$3x^2 - 6xy - 15y^2 = -2?$$

89. Equations with imaginary roots. The student should now be able to solve and to check the solution of an equation which has imaginary roots.

EXERCISES

Solve the equations which follow, and check the results:

1. $x^2 + 2x + 4 = 0$.
2. $x^2 - 6x + 12 = 0$.
3. $x^2 + 5x + 9 = 0$.
4. $x^2 - 3x + 15 = 0$.
5. $2x^2 + 3x + 4 = 0$.
6. $x^2 + x + 1 = 0$.
7. $x^2 - x + 1 = 0$.
8. $6x^2 - 8x + 15 = 0$.
9. $7x^2 + 12x + 8 = 0$.
10. $3x^2 + 8x + 10 = 0$.

11. $x^3 = 1$.

HINT.
If $x^3 = 1$,
then $x^3 - 1 = 0$,
and $(x - 1)(x^2 + x + 1) = 0$,
or $x - 1 = 0$,
and $x^2 + x + 1 = 0$.

12. $x^3 = -1$.
13. $x^3 = 27$.
14. $x^3 = -8$.
15. $x^4 = 1$.
16. $x^4 = 25$.
17. $x^6 = 1$.
18. $x^6 = 64$.
19. $x^3 = 125$.
20. $x^3 = -64$.

21. How many square roots has any real number? cube roots? fourth roots? sixth roots?

22. What do the preceding exercises suggest regarding the number of nth roots which any number has?

23. $8x^3 - 1 = 0$.

24. $27x^3 + 64 = 0$.

25. $x^4 - 4x^2 - 21 = 0$.

26. $x^4 - x^3 - 3x^2 + 3x = 0$.

27. $x^4 + 5x^2 - 6 = 0$.

28. $3x^4 + 10x^2 + 3 = 0$.

29. $6x^4 - 7x^2 - 3 = 0$.

30. $2x^4 + 5x^2 + 2 = 0$.

31. $15x^4 + 17x^2 + 4 = 0$.

32. $(x^2 + 9)(x^2 + 5x + 8) = 0$.

33. $(x^2 + 3x)^2 - 2(x^2 + 3x) - 8 = 0$.

34. $(2x^2 + 3x)^2 - 41(2x^2 + 3x) + 378 = 0$.

NOTE. Long before the time of Gauss, mathematicians had performed the operations of multiplication and division on complex numbers by the same rules that they used for real numbers. As early as 1545 Cardan showed that the product of $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ was the real number 40. However, he was not always equally fortunate in obtaining correct results, for in another place he sets $\frac{1}{4}(-\sqrt{-\frac{1}{4}}) = \frac{1}{\sqrt{64}} = \frac{1}{8}$.

Even the rather complicated formula for extracting any root of a complex number was discovered in the early part of the eighteenth century. But all these operations were purely formal, and seemed to most mathematicians a mere juggling with symbols until Gauss showed clearly the place and usefulness of such numbers.

NOTE ON THE USE OF IMAGINARIES. We have explained the laws of addition, subtraction, multiplication, and division for imaginary (and complex) numbers and have made some use of them. It is largely because imaginaries obey these laws that we call them numbers, for it must be admitted that we cannot count objects with imaginary numbers. Nor can we state by means of them our age, our weight, or the area of the earth's surface. It should be remembered, however, that we can do none of these things with negative numbers. We may

have a group of objects — books, for example — whose number is 5; but no group of *objects* exists whose number is -5, or -3, or any negative number whatever. If it is asked, How, then, can negative numbers and imaginary numbers have any practical use? the answer is this: They have a practical use because when they enter into our calculations and we have performed the necessary operations upon them and obtained our final result, that result can frequently be interpreted as a concrete number like those dealt with in ordinary arithmetic. Moreover, if the result cannot be so interpreted, it is, in applied mathematics at least, finally rejected.

In that part of electrical engineering where the theory and measurement of alternating currents are treated, complex numbers have had extensive use. Their use in the difficult problems which there arise has given a briefer, a more direct, and a more general treatment than where such numbers are not used.

In theoretical mathematics complex numbers have been of great value in many ways. For example, numerous important theorems about functions are more easily proved under the assumption that the variable is complex. Then, by letting the imaginary part of the complex number become zero, we obtain the proof of the theorem for real values of the variable. Indeed, the student need not go very far beyond this point in his mathematical work to learn that, if e is 2.7182^{+} (see page 238), $e^{\sqrt{-1}} + e^{-\sqrt{-1}}$ is equal to the real number 1.082^{+}. At the same time he will learn also how such a form arises, and something of its importance. In a way which we cannot now explain, even so involved an expression as $(a + ib)^{c+id}$ has in higher work a meaning and a use. If the student pursues his mathematical studies far enough, that meaning and use and a multitude of other uses for complex numbers will become familiar to him. But the numbers which we have learned in this book to use, namely fractions, negative numbers, irrational numbers, and complex numbers, complete the number system of ordinary algebra, for it can be proved that from the fundamental operations no other forms of number can arise.

CHAPTER XII

FUNCTIONS AND THEIR GRAPHS

90. Functions. The notion of function is one of the central concepts in modern mathematics. The basic idea involved is that of the dependence of one quantity on one or more other quantities. Countless functional relations exist in our everyday affairs, most of which are too complicated for mathematical expression. For example, the height of a boy is a function of his age, his health, his heredity, and many other elements. But it would be impossible to measure exactly the effect of health or heredity on the height of a boy. Hence this function is not one that can be dealt with mathematically. On the other hand, the distance that a body falls from a state of rest under the influence of the force of gravity can be measured and expressed by the formula $s = \frac{1}{2} gt^2$, in which s equals the distance, g equals 32, and t equals the elapsed time in seconds. In fact the formula can be derived theoretically from certain laws of nature. Consequently, this function is one that can be used mathematically. In general one variable is a function of another if it depends on the other for its value. Thus, in $x = y + 5$, x is a function of y or vice versa.

In the study of algebra we are concerned with the mathematical formulas in terms of which the physical relations are expressed, rather than with the actual phenomena themselves. Consequently in this chapter we

shall content ourselves with a study of the graphical representation of equations that actually express many of the relationships which we know in our everyday life.

91. Names of functions. A function is called **linear, quadratic,** or **cubic** according as its degree with respect to the unknown or unknowns is first, second, or third respectively.

Thus $3x - 5$ is a linear function of x; $2x^2 - 5x + 8$ is a quadratic function of x; and $y^3 - 2y^2 + y - 10$ is a cubic function of y.

In the study of functions the unknown is often called the variable, since from this point of view the problem is not so much the finding of an unknown as it is the study of the changes of a variable quantity.

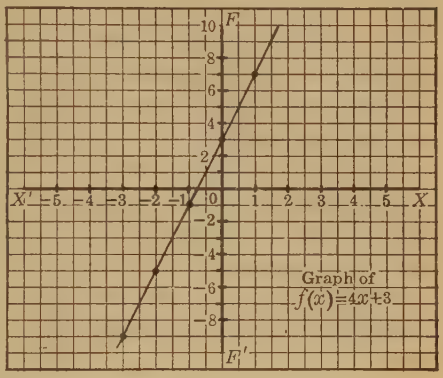
92. Notation for a function. After a function of any variable x has once been given it is usual to refer to it later in the same discussion by the symbol $f(x)$, which is read **the function of** x, or more briefly, f **of** x.

93. Linear functions. The expression $4x + 3$ is a function of x, and the value of this binomial varies with x. The following table gives a partial view of the relative change of values between x and the function $4x + 3$:

If $x =$	-4	-3	-2	-1	0	1	2	3	4
then $f(x) = 4x + 3 =$	-13	-9	-5	-1	3	7	11	15	19

This relation can be represented graphically by using the same x-axis as before (section 49) and using the y-axis as the function axis; that is, laying off values of x horizontally and corresponding values of the function $4x + 3$

vertically. The graph resulting from the above table of values is shown in the accompanying figure. It can be shown that the graph of a linear function is always a **straight line.**



EXERCISES

1. Construct the graph of the function $2x + 3$.

2. Construct the graph of the function $5x - 1$.

3. Construct the graph of the function $3x + 7$.

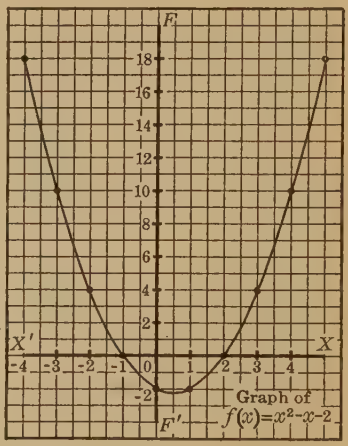
4. The relation between the readings on the Fahrenheit thermometer scale and those on the Centigrade scale is expressed by the equation $F = \frac{9}{5}C + 32$. Construct the graph for $f(C) = \frac{9}{5}C + 32$. What is the Fahrenheit reading when the temperature is 50° Centigrade? 0°? 100°?

5. An automobile starts to ascend a grade at a speed of 30 miles per hour, but each second slows up 2 miles per hour. The function of the time which gives the speed of the car is $V = f(t) = 30 - 2t$. Construct the graph of the function $f(t) = 30 - 2t$. Read from the graph the speed of the car after 5 seconds, after 15 seconds, after 20 seconds.

94. Quadratic functions. The function $f(x) = x^2 - x - 2$ may be represented graphically as follows:

If $x =$	-4	-3	-2	-1	0	1	2	3	4
then $f(x) = x^2 - x - 2 =$	18	10	4	0	-2	-2	0	4	10

Plotting the points corresponding to the numbers in the table we obtain the accompanying graph.



The graph of a quadratic function in one variable is a curve called a **parabola.** It may be sharper or flatter than the accompanying graph, but is always of the same general shape.

EXERCISES

Construct the graphs of the following:

1. $f(x) = x^2 + 2x - 5$.
2. $f(x) = 4x^2 - 3x + 1$.
3. $f(x) = 2x^2 - 2x - 3$.
4. $f(x) = x^2 - 7x + 6$.
5. $f(x) = 3x^2 + 5x - 10$.
6. $f(x) = 5x^2 - 3x + 5$.

7. A body falling from rest under the influence of gravity follows the law $s = f(t) = 16t^2$, in which s is the distance fallen in feet, and t is the time, in seconds, elapsed since the body started falling. Construct the graph of $f(t) = 16t^2$.

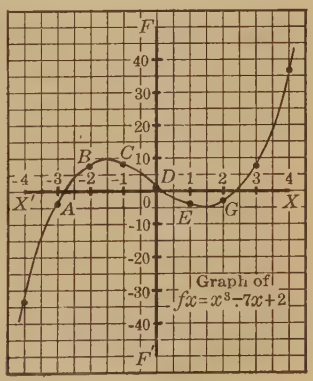
NOTE. In the study of analytical geometry one takes up systematically the curves which represent equations of the various degrees, beginning with the simplest. It turns out, as we have already seen, that the linear equation is represented by a straight line. Equations of the second degree in x and y lead to the so-called "conic sections."

One of the most interesting and important aspects of the graphical method is the fact that the simplest equations correspond to the most useful curves both in pure science and in nature. Among the commonest curves in nature are the circle and the parabola. Their equations are the very simplest equations of the second degree.

95. Graph of a cubic function. The graph of a cubic function is obtained in the same general way as that of a quadratic function. The function $x^3 - 7x + 2$ may be represented graphically by proceeding as follows:

If $x =$	-4	-3	-2	-1	0	1	2	3	4
then $f(x) = x^3 - 7x + 2 =$	-34	-4	8	9	2	-4	-4	8	38

Plotting the points corresponding to the numbers in table (as in the figure below), we obtain the points $A(-3, -4)$, B, C, D, E, and G, in the order named. The curve crosses the x-axis three times: once between 2 and 3, again between 0 and 1, and a third time between -2 and -3. Above G the curve rises indefinitely, and below A it falls indefinitely. In each case it becomes more and more nearly straight as it recedes from the x-axis, never crossing either axis again.



In forming a table of values two pairs are sufficient for a linear function, but more are needed for a quadratic function and still more for a cubic function. Usually the higher the degree of the function, the more points are needed in constructing its graph. It should be noted that in making a good graph the

number of points is not so important as is their distribution, which should be such as faithfully to outline the entire curve. Where the graph curves rapidly or makes sharp turns the points should be close together. Such places are difficult to locate before the graph is constructed; hence one should make a table of values which appears to be sufficient and plot them. Then inspection of the plotted points will usually show where sharp turns or rapid curvature exists. The table of values should then be properly extended and the additional points located. Repetition of this last step will enable one to draw a graph which accurately pictures the variation of the function.

It should be observed here that scales on the two axes need not be the same. Some experience is required to choose for the two axes the scales which are best suited to bring out clearly the shape of the curve. In general the graph should be drawn to *as large* a scale, *in both directions*, as the size of the paper permits. What that will be for each axis can be decided by inspecting the table of values. For example, when the dimensions of the preceding graph are once determined one can see from the table that all values of x are easily represented but that it may be undesirable to try to represent the values of the function less than -34 or greater than 38.

EXERCISES

Construct the graphs of the following:

1. $f(x) = x^3 + 2x - 1$.
2. $f(x) = x^3 - 5x - 3$.
3. $f(x) = x^3 - 3x + 1$.
4. $f(x) = x^4 - 12x^2 + 24$.

NOTE. The notion of a function is one of the three or four most fundamental ideas in modern mathematics. Only the simplest examples are given in this book, but many others involving expressions of the utmost complexity have been studied by mathematicians for many years. An important reason for the study of functions is found in the fact that all kinds of facts and principles which we meet in the study of nature can be expressed symbolically by means of functions, and the discovery of the properties of such functions helps us to understand the meaning of the facts. A complete understanding of

the laws of falling bodies, light, electricity, or sound could never be reached without the study of the mathematical functions which these phenomena suggest.

When the electrician, the architect, or the artillerist meets a problem, he frequently must represent quantities by letters. The x and the y may represent the measures of objects in nature, but the solution has become merely an operation of algebra. As students of algebra we are not concerned with the origin of the function or expression, but merely with the numeric determination of some unknown or the simplification of some expression.

96. Graphical solution of equations in one unknown. In elementary algebra one of the most important applications of the preceding sections is their use in solving equations in one unknown. The ideas involved can be made clear by questions on the graphs of sections 93, 94, and 95.

ORAL EXERCISES

1. From the graph in section 93, determine the value of the function $f(x) = 4x + 3$ at the point where its graph crosses the x-axis.

2. Does the value of x for this point satisfy the equation $4x + 3 = 0$?

3. Solve $4x + 3 = 0$ without reference to the graph.

4. What point on the graph represents the root of $4x + 3 = 0$?

5. From the graph in section 94, determine the values of the function $f(x) = x^2 - x - 2$ at the points where its graph crosses the x-axis.

6. Do the values of x for these points satisfy the equation $x^2 - x - 2 = 0$?

7. Solve $x^2 - x - 2 = 0$ without referring to the graph.

8. What points on the graph represent the roots of $x^2 - x - 2 = 0$?

9. From the graph in section 95, determine the values of the function $x^3 - 7x + 2$ at the points where its graph crosses the x-axis.

10. Do these values of x make the function $x^3 - 7x + 2$ equal zero?

11. What method could be used to solve the equation $x^3 - 7x + 2 = 0$?

97. The process of graphical solution. From what precedes, it is apparent that the steps in the graphical solution of an equation in one unknown are:

I. *Transpose the terms so that the right member is zero.*
II. *Graph the function in the left member.*
III. *The values of* x *for the points where the graph crosses the* x*-axis are the real roots of the equation.*

The algebraic solutions of a linear equation and a quadratic equation in one unknown are so simple that except for the purpose of illustration their graphical solution is comparatively unimportant. The algebraic solution of cubic and higher equations, except in simple cases, is much more difficult than the solution of the quadratic and is never presented in an elementary course. For this reason and for the insight it gives into equations in general, the *graphical* solution of the cubic and higher equations is important and illuminating. For such equations if the method of factoring fails, the graphical method is the only method open to the student at this point in his progress.

EXERCISES

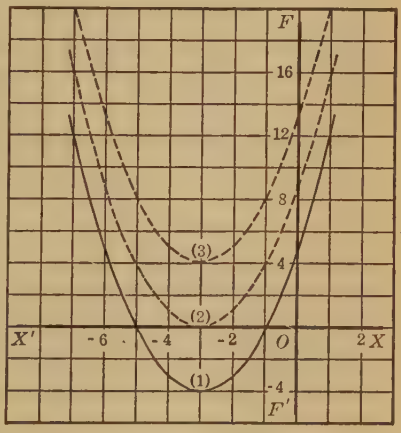
Solve graphically obtaining results to the first decimal place:

1. $x^2 + 2x - 1 = 0$.
2. $x^3 - 10x + 5 = 0$.
3. $x^3 - 8x - 10 = 0$.
4. $x^3 - 6x = 0$.
5. $x^3 - 3x^2 + 1 = 0$.
6. $x^4 - 12x^2 + 15 = 0$.

98. Imaginary roots. An equation of the second or a higher degree often has **imaginary roots.** Such roots cannot be obtained by the graphical methods so far considered. A study of the graphs which follow will make clear why this is true.

Consider the following equations:

$$x^2 + 6x + 5 = 0, \quad (1)$$
$$x^2 + 6x + 9 = 0, \quad (2)$$
$$x^2 + 6x + 13 = 0. \quad (3)$$



The graphs of the functions in the left members of equations (1), (2), and (3) are given in the accompanying figure. The three functions differ only in their constant terms, for 4 added to the constant term of (1) gives the constant term of (2), and 4 added to the constant term of (2) gives the constant term of (3). Apparently, as the constant term is increased the graph rises without change of shape and without motion to the left or to the right.

From the graph the roots of $x^2 + 6x + 5 = 0$ are seen to be -5 and -1. These results are obtained from factoring; $x^2 + 6x + 5 = 0$, or $(x + 5)(x + 1) = 0$. Whence $x = -5$ or -1.

If we imagine curve (1) to move upward, the two roots change in value and become the single root of curve (2), which touches the x-axis at a point where x equals -3. Solving $x^2 + 6x + 9 = 0$ by factoring gives $(x + 3)(x + 3) = 0$. Whence $x = -3$.

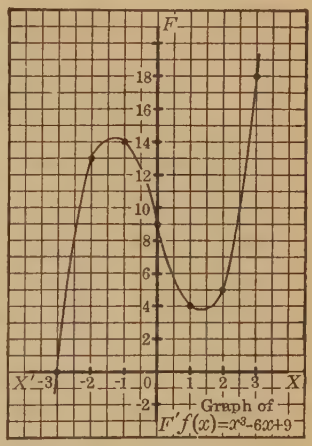
If we now imagine curve (1) to move still farther upward from its position (2), it will no longer cut the x-axis. Therefore, when the curve reaches the position of (3), it does not cut the x-axis at all, and hence cannot be expected to show the values of the roots of the equation $x^2 + 6x + 13 = 0$, as, in fact, it does not. The graph does show, however, that the value of $x^2 + 6x + 13$ at the lowest point of the curve is 4. This means that for every real value of x, positive or negative, $x^2 + 6x + 13$ is never less than 4. The graph of (3) makes clear that no real number if substituted for x will make $x^2 + 6x + 13$ equal zero.

It can be shown by the method of section 82 that the roots of $x^2 + 6x + 13 = 0$ are $-3 + 2\sqrt{-1}$ and $-3 - 2\sqrt{-1}$.

NOTE. It required the genius of no less a man than Sir Isaac Newton first to observe from the graph of a function that two of its roots become imaginary simultaneously. He also saw that an equation with two of its roots equal to each other is, in a certain sense, the limiting case between equations in which the corresponding roots appear as two real and distinct roots and those in which they appear as imaginary roots.

99. Imaginary roots for a cubic equation. If we attempt to solve $x^3 - 6x + 9 = 0$ graphically, we obtain the

graph of the accompanying figure. The curve crosses the x-axis at $x = -3$. This is the only real root the equation has; the other two are imaginary. The roots can here be obtained by factoring; thus $(x + 3)(x^2 - 3x + 3) = 0$. The roots of $x^2 - 3x + 3 = 0$, $\frac{3 \pm \sqrt{-3}}{2}$, are the two imaginary roots of the *cubic* equation.



EXERCISES

As far as possible solve graphically, finding roots to one decimal place:

1. $x^3 - x^2 - x - 2 = 0$.
2. $x^3 + x^2 + x + 1 = 0$.
3. $x^3 - 7x - 6 = 0$.
4. $x^3 - 3x^2 + 5x - 15 = 0$.
5. $x^3 + 2x^2 - 11x + 20 = 0$.
6. $x^3 + 2x^2 - 1 = 0$.
7. $x^3 + 4x^2 - 8x + 24 = 0$.
8. $x^4 - 16 = 0$.
9. $x^4 - 9x^2 + x + 8 = 0$.
10. $x^4 - 8x^2 + 5 = 0$..

CHAPTER XIII

THEORY OF QUADRATIC EQUATIONS

100. Formation of equations with given roots. According to section 39, the equation $(x - 4)(x - 1) = 0$ has the roots 4 and 1. In general, the equation $(x - r_1)(x - r_2) = 0$ has the roots r_1 and r_2, because either of these numbers, when substituted for x, satisfies the equation. Hence we can always find an equation whose roots are two given numbers r_1 and r_2 by setting the product of the binomials $x - r_1$ and $x - r_2$ equal to zero.

For example, an equation whose roots are 5 and 8 is seen in $(x - 5)(x - 8) = 0$, or $x^2 - 13x + 40 = 0$.

EXERCISES

Form equations whose roots are the following:

1. 3, 4.
2. 1, 5.
3. 2, $-$ 1.
4. $-$ 3, $-$ 5.
5. 3, $\frac{1}{2}$.
6. 7, $\frac{3}{4}$.
7. $-\frac{2}{5}, \frac{3}{7}$.
8. $-\frac{3}{10}, -\frac{2}{7}$.
9. $3 + \sqrt{2}, 3 - \sqrt{2}$.
10. $2 \pm \sqrt{5}$.
11. $1 - \sqrt{-3}, 1 + \sqrt{-3}$.
12. $-8 + \sqrt{-5}, -8 - \sqrt{-5}$.
13. $\frac{1}{3} + \sqrt{\frac{4}{3}}, \frac{1}{3} - \sqrt{\frac{4}{3}}$.
14. $\frac{1}{4} + \sqrt{-\frac{3}{2}}, \frac{1}{4} - \sqrt{-\frac{3}{2}}$.

15. 3, 6, 8.

HINT. $(x-3)(x-6)(x-8)=0$.

16. 2, 7, 3.

17. 2, -3, 8.

18. $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}$.

19. $\sqrt{-3}, -\sqrt{-3}, 3$.

20. $2+\sqrt{5}, 2-\sqrt{5}, 6$.

21. $2, -2, \sqrt{-1}, -\sqrt{-1}$.

22. $\sqrt{3}-\sqrt{5}, \sqrt{3}+\sqrt{5}, -\sqrt{3}-\sqrt{5}, -\sqrt{3}+\sqrt{5}$.

101. Relations between roots and coefficients of $x^2+bx+c=0$. By direct multiplication we obtain from

$$(x-r_1)(x-r_2)=0 \tag{1}$$

the equation $x^2-(r_1+r_2)x+r_1r_2=0$. (2)

Since r_1 and r_2 are the roots of (1), it appears from an inspection of (2) that the quadratic equation

$$x^2+bx+c=0$$

has the roots r_1 and r_2, provided $b=-(r_1+r_2)$ and $c=r_1r_2$.

For example, we may form at once the equation whose roots are 3 and 5, as follows:

$$x^2-(3+5)x+3\cdot 5=0, \text{ or } x^2-8x+15=0.$$

Similarly for the cubic equation $x^3+bx^2+cx+d=0$, roots are r_1, r_2, and r_3, we have $(x-r_1)(x-r_2)(x-r_3)=0$.

Then $b=-(r_1+r_2+r_3)$,

$c=r_1r_2+r_1r_3+r_2r_3$,

and $d=-r_1r_2r_3$.

EXERCISES

Form equations whose roots are the following:

1. 3, 8. 2. 4, 9. 3. $-2, 5$.

4. $-2, -3$.
5. $-8, -1$.
6. $2, 3, -4$.
7. $1, -2, 5$.
8. $\sqrt{2}, -\sqrt{2}$.
9. $3\sqrt{5}, -3\sqrt{5}$.
10. $2+\sqrt{3}, 2-\sqrt{3}$.
11. $1+\sqrt{7}, 1-\sqrt{7}$.

102. Relation between roots and coefficients of $ax^2 + bx + c = 0$. We will now show the relations which exist between the roots and the coefficients of the *general quadratic equation* $ax^2 + bx + c = 0$. By section 82 the roots of $ax^2 + bx + c = 0$ are

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \text{ and } r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Adding r_1 and r_2, we have

$$r_1 + r_2 = \frac{-b + \sqrt{b^2 - 4ac} - b - \sqrt{b^2 - 4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}.$$

Therefore $$-(r_1 + r_2) = \frac{b}{a}.$$

Multiplying r_1 by r_2, we have

$$r_1 r_2 = \frac{-b + \sqrt{b^2 - 4ac}}{2a} \cdot \frac{-b - \sqrt{b^2 - 4ac}}{2a}$$

$$= \frac{b^2 - (b^2 - 4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Therefore $$r_1 r_2 = \frac{c}{a}.$$

These results may be expressed verbally as follows:
For the equation $ax^2 + bx + c = 0$,

I. *The sum of the roots with its sign changed is* $\frac{b}{a}$.

II. *The product of the roots is* $\frac{c}{a}$.

These relations frequently afford the simplest means of checking the solution of a quadratic equation, as illustrated in Exercises 18–25, below.

EXERCISES

Form equations whose roots are the following:

1. $\frac{4}{3}, -\frac{3}{2}$.

Solution.

$$-\left(\frac{4}{3} - \frac{3}{2}\right) = \frac{1}{6} = \frac{b}{a}.$$

$$\left(\frac{4}{3}\right)\left(-\frac{3}{2}\right) = -2 = \frac{c}{a}.$$

Hence the required equation is

$$x^2 + \tfrac{1}{6}x - 2 = 0$$

or

$$6x^2 + x - 12 = 0.$$

2. $\frac{8}{3}, \frac{1}{4}$.
3. $.5, -.75$.
4. $1\frac{2}{3}, -4\frac{1}{2}$.
5. $4.75, 5.25$.
6. $3 + 2\sqrt{2}, 3 - 2\sqrt{2}$.
7. $9 + \sqrt{-3}, 9 - \sqrt{-3}$.
8. $\frac{4}{3} + \sqrt{6}, \frac{4}{3} - \sqrt{6}$.
9. $\frac{1}{4} + \frac{1}{4}\sqrt{-7}, \frac{1}{4} - \frac{1}{4}\sqrt{-7}$.
10. $1\frac{7}{10} \pm \sqrt{10}$.
11. $\pm\sqrt{-1} + 2$.
12. $-b, \frac{1}{b}$.
13. $\frac{2a}{3}, \frac{5a}{3}$.
14. $a + 1, a - 1$.
15. $\frac{m+n}{m-n}, \frac{m-n}{m+n}$.
16. $3, 7, \frac{1}{3}$.
17. $5, -5, \frac{1}{5}$.

Solve the following equations by the use of the formula and check by the use of I and II above:

18. $x^2 - 5x + 4 = 0$.
19. $x^2 - 2x - 5 = 0$.
20. $x^2 - x - 7 = 0$.
21. $x^2 - 5x - 10 = 0$.
22. $x^2 + 2x + 1 = 0$.
23. $x^2 + 4x + 4 = 0$.
24. $x^2 + 3x + 3 = 0$.
25. $3x^2 + 2x - 7 = 0$.

Find the value of the literal coefficient in the following:

26. $x^2 - 4x + a = 0$, if one root is 3.

HINT. The sum of the roots = 4.

27. $x^2 - 2x - c = 0$, if one root is 4.
28. $x^2 + cx - 15 = 0$, if one root is 5.
29. $x^2 + 2bx + 10 = 0$, if one root is -2.
30. $x^2 + 6x + a = 0$, if one root is twice the other.
31. $x^2 + 5x - c = 0$, if one root exceeds the other by 3.
32. $x^2 - 3x + b = 0$, if the difference between the roots is 7.

103. Character of the roots of a quadratic equation. It is often desirable to determine whether the roots of a given quadratic equation are **real or imaginary, rational or irrational, equal or unequal**, without solving the equation. This can be accomplished by use of the formulas for the roots of the quadratic $ax^2 + bx + c = 0$:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{1}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

These expressions are seen to differ from each other only in the sign preceding the radical. The expression

$$b^2 - 4ac,$$

which appears under the radical sign, is called the **discriminant** of the quadratic. The only way in which r_1 or r_2 can be a complex number is for the discriminant to be negative, since this is the only part of the root under a radical sign. If all of the coefficients a, b, and c are rational, r_1 or r_2 can be rational only when $\sqrt{b^2 - 4ac}$ is rational; that is, when the discriminant is a perfect square.

Hence if a, b, and c are rational, an inspection of (1) and (2) shows that the following statements are true:

I. *If $b^2 - 4ac$ is positive and not a perfect square, the roots are real, unequal, and irrational.*

For example, in $x^2 - 6x + 2 = 0$ the discriminant $b^2 - 4ac$ equals $(-6)^2 - 4 \cdot 1 \cdot 2 = 28$, which is not a perfect square. The roots of the equation are the real, unequal, and irrational numbers $3 \pm \sqrt{7}$.

II. *If $b^2 - 4ac$ is positive and a perfect square, the roots are real, unequal, and rational.*

For example, in the equation $3x^2 - 5x + 2 = 0$ the discriminant $b^2 - 4ac$ equals $(-5)^2 - 4 \cdot 3 \cdot 2 = 1$, which is a perfect square. The roots are the real, unequal, rational numbers $\frac{2}{3}$ and 1.

III. *If $b^2 - 4ac$ is zero, the roots are real and equal.*

For example, in the equation $9x^2 - 12x + 4 = 0$, the discriminant $b^2 - 4ac$ equals $(-12)^2 - 4 \cdot 9 \cdot 4 = 144 - 144 = 0$. Here the only number which satisfies this equation is $\frac{2}{3}$, so in one sense the equation has only one root. But since

the left member has two identical factors each of which affords the same root of the equation, it is customary to say that the equation has two equal roots.

IV. *If* $b^2 - 4\,ac$ *is negative, the roots are imaginary.*

For example, in the equation $3x^2 - 7x + 5 = 0$ the discriminant $b^2 - 4\,ac$ equals $(-7)^2 - 4 \cdot 3 \cdot 5 = 49 - 60 = -11$. The roots of the equation are the conjugate imaginaries $\frac{7 + \sqrt{-11}}{6}$ and $\frac{7 - \sqrt{-11}}{6}$.

ORAL EXERCISES

Through the use of the discriminant determine the character of the roots of the following equations:

1. $x^2 + 2x + 1 = 0$.
2. $x^2 - 3x + 2 = 0$.
3. $2x^2 - 5x - 3 = 0$.
4. $4x^2 + 3x - 7 = 0$.
5. $7x^2 - 5x + 3 = 0$.
6. $2x^2 - 2x + 1 = 0$.
7. $x^2 - x + 1 = 0$.
8. $2x^2 - 5x + 7 = 0$.
9. $5x^2 + 5x - 1 = 0$.
10. $3x^2 - 4x + 5 = 0$.
11. $5x^2 + 6x - 12 = 0$.
12. $18x^2 + 36x + 25 = 0$.
13. $6x^2 + 11x + 3 = 0$.
14. $6x^2 + 9x - 60 = 0$.

EXERCISES

Determine the value of k which will make the roots of the following equations equal:

1. $x^2 + kx + 25 = 0$.

Solution. $a = 1, b = k, c = 25$.

Hence $b^2 - 4\,ac = k^2 - 100$.

For the roots to be equal $b^2 - 4\,ac$ must equal zero.

Therefore $k^2 - 100 = 0.$

Whence $k = \pm 10.$

Check. Substituting 10 for k in the original equation,

$$x^2 + 10x + 25 = (x + 5)^2 = 0.$$

Whence $x = -5$ only.

Similarly, substituting -10 for k,

we obtain $x^2 - 10x + 25 = (x - 5)^2 = 0.$

Whence $x = +5$ only.

2. $x^2 - kx + 4 = 0.$
3. $x^2 + 16x + k = 0.$
4. $4x^2 + 8x + k = 0.$
5. $x^2 - 4kx + 64 = 0.$
6. $2x^2 - kx + 2 = 0.$
7. $4x^2 + 12x + k - 1 = 0.$
8. $kx^2 + 42x + 49 = 0.$
9. $kx^2 - 36x + 36 = 0.$
10. $36x^2 - (k + 5)x + 25 = 0.$
11. $(k^2 + 13)x^2 + 112x + 64 = 0.$

CHAPTER XIV

GRAPHS OF QUADRATIC EQUATIONS IN TWO VARIABLES

104. Graph of a quadratic equation in two variables. Before solving graphically a quadratic system, the method of graphing *one* quadratic equation in *two* variables must be clearly understood.

Examples

1. Construct the graph of $x^2 = 5y$.

Solution. Solving for y, we get

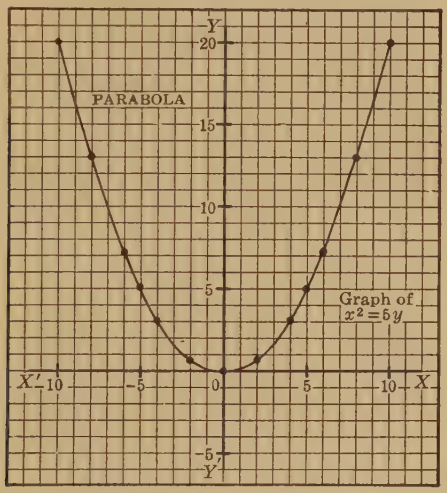
$$y = \frac{x^2}{5}.$$

We now assign values to x and then compute the approximate corresponding values of y. Tabulating the results gives:

If $x =$	10	8	6	5	4	3	2	1	0	-2	-4	-5	-6	-8	-10
Then $y =$	20	$12\frac{4}{5}$	$7\frac{1}{5}$	5	$3\frac{1}{5}$	$1\frac{4}{5}$	$\frac{4}{5}$	$\frac{1}{5}$	0	$\frac{4}{5}$	$3\frac{1}{5}$	5	$7\frac{1}{5}$	$12\frac{4}{5}$	20

Using an x-axis and a y-axis as in graphing linear equations, plotting the points corresponding to the real numbers in the table, and drawing the curve determined by these points, we obtain the graph of the adjacent figure. The curve is called a **parabola.** The parabola, along with certain other graphs treated in this chapter, belongs to a family of curves known as **conic sections.** These will be discussed in more detail later.

The graph of any equation of the form $x^2 = ay$ is a parabola. This is a special case of the quadratic function considered in section 94.



2. Graph the equation $xy + 3 = 0$.

Solution. Solving for y, $y = -\frac{3}{x}$.

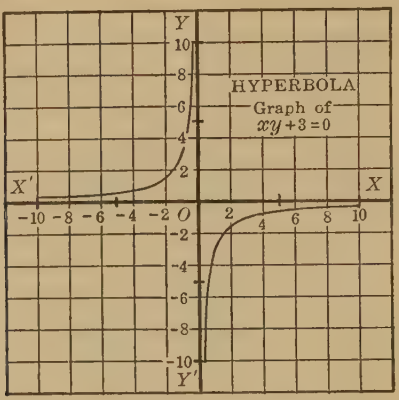
Assigning values to x as indicated in the following table, we then compute the corresponding values of y:

If $x=$	-10	-8	-6	-4	-3	-2	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{2}$	$\frac{3}{4}$	1	2	3	4	6	8	10
Then $y=$	$\frac{3}{10}$	$\frac{3}{8}$	$\frac{1}{2}$	$\frac{3}{4}$	1	$1\frac{1}{2}$	3	4	6	12	-12	-6	-4	-3	$-1\frac{1}{2}$	-1	$-\frac{3}{4}$	$-\frac{1}{2}$	$-\frac{3}{8}$	$-\frac{3}{10}$

Proceeding as before with the numbers in the table, we obtain the two-branched curve of the figure below, which does not touch either axis. The curve is called a hyperbola.

The graph of any equation of the form $xy = K$ is a *hyperbola*. The curve for $xy = K$ (K = any constant) is

always in the same general position; that is, if K is positive, one branch of the curve lies in the first quadrant and



the other branch in the third. If K is negative, one branch lies in the second quadrant and the other in the fourth.

3. Graph the equation $x^2 + y^2 = 25$.

Solution. Solving for y, $y = \pm\sqrt{25 - x^2}$.

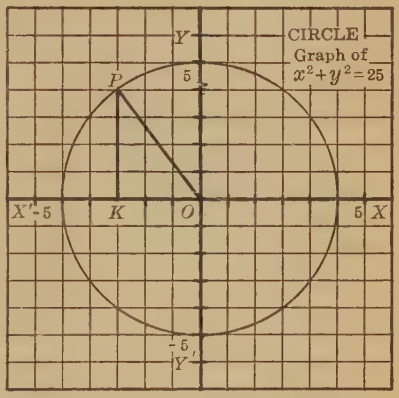
Assigning values to x as indicated in the following table, we obtain from page 330 the corresponding values of y:

If $x=$	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
Then $y=$	$\pm\sqrt{-11}$	0	±3	±4	±4.58	±4.90	±5	±4.90	±4.58	±4	±3	0	$\pm\sqrt{-11}$

For values of x numerically greater than 5 it appears that y is imaginary. The points corresponding to the pairs of real numbers in the table lie on the circle in the accompanying figure. The center of the circle is at the origin, and the radius is 5.

The graph of any equation of the form $x^2 + y^2 = r^2$ is a circle whose radius is r. This can be proved from the

right triangle PKO. If P represents *any* point on the circle, OK equals the x-distance of P, KP equals the y-distance, and OP equals the radius. Now $\overline{OK}^2 + \overline{KP}^2 = \overline{OP}^2$; that is, $x^2 + y^2 = r^2$. It follows, then, that the graphs of $x^2 + y^2 = 16$ and $x^2 + y^2 = 10$ are circles whose centers are at the origin and whose radii are 4 and $\sqrt{10}$ respectively. Hereafter, when it is required to graph an equation of the form $x^2 + y^2 = r^2$, the student may use compasses and, with the origin as center and with the proper radius (the square root of the constant term), describe the circle at once.



In all the graphical work which follows, the student will save time by obtaining from the table on page 330 the square roots or cube roots which he may need.

4. Graph the equation $9x^2 + 4y^2 = 36$.

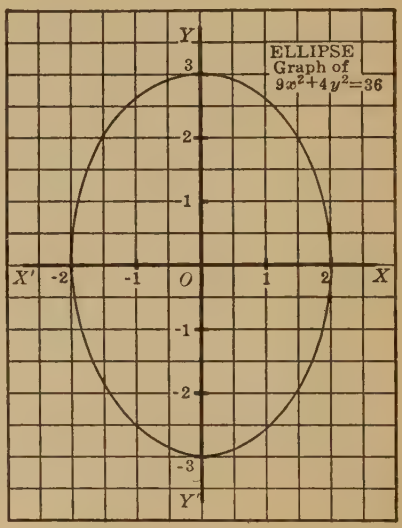
Solution. Solving for y, $y = \pm\sqrt{\dfrac{36 - 9x^2}{4}}$

$$= \pm \tfrac{1}{2}\sqrt{36 - 9x^2}$$
$$= \pm \tfrac{3}{2}\sqrt{4 - x^2}.$$

Proceed as in Example 3:

If $x =$	-3	-2	$-1\frac{1}{2}$	-1	$-\frac{1}{2}$	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3
Then $y =$	$\pm\frac{3}{2}\sqrt{-5}$	0	± 1.98	± 2.60	± 2.90	± 3	± 2.90	± 2.60	± 1.98	0	$\pm\frac{3}{2}\sqrt{-5}$

For any value of x numerically greater than 2, y is imaginary. The points corresponding to the real numbers in the table lie on the graph of the adjacent figure. The curve is called an ellipse.



The graph of any equation of the form of $ax^2 + by^2 = c$ in which a and b are unequal and of the same sign as c is an *ellipse*.

NOTE. These three curves — the ellipse, the hyperbola, and the parabola — were first studied by the Greeks, who proved that they are the sections which one obtains by cutting a cone by a plane. Not for hundreds of years afterwards did anyone imagine that these curves actually appear in nature, for the Greeks regarded them merely as geometric figures, and not at all as curves that have anything to do with our everyday life. One of the most important discoveries of astronomy was made by Kepler (1571–1630), who showed that the earth revolves around the sun in an ellipse, and stated the laws which govern the motion. Those comets that return to our field of vision periodically also have elliptic orbits, while those that appear once, never to be seen again, describe parabolic or hyperbolic paths.

The path of a ball thrown through the air in any direction, except vertically upward or downward, is a parabola. The approximate parabola which a projectile actually describes depends on the elevation of the gun (the angle with the horizontal), the quality of the powder, the amount of the charge, the direction of the wind, and various other conditions. This makes gunnery a complex subject.

EXERCISES

Construct the graphs of the following equations, and state the name of each curve obtained:

1. $x^2 = 6y$.
2. $y^2 + 3x = 0$.
3. $x^2 + y^2 = 4$.
4. $x^2 + y^2 = 15$.
5. $x^2 - y^2 = 36$.
6. $xy = 10$.
7. $xy = -8$.
8. $16x^2 + 25y^2 = 400$.
9. $9x^2 - 4y^2 = 36$.
10. $16x^2 + 9y^2 = 144$.

105. Graphical Solution of Quadratic Systems. That we may solve a system of two quadratic equations by a method similar to that employed in section 49 for linear equations appears from the

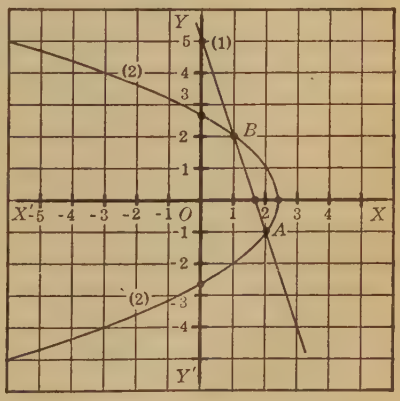
EXAMPLES

1. Solve graphically $\begin{cases} 3x + y = 5, & (1) \\ y^2 + 3x = 7. & (2) \end{cases}$

Solution. Constructing the graphs of (1) and (2), we obtain the straight line and the parabola shown in the adjacent figure. There are two sets of roots corresponding to two points of intersection, which are

$$A \begin{cases} x = 2, \\ y = -1 \end{cases}$$

and $$B \begin{cases} x = 1, \\ y = 2. \end{cases}$$



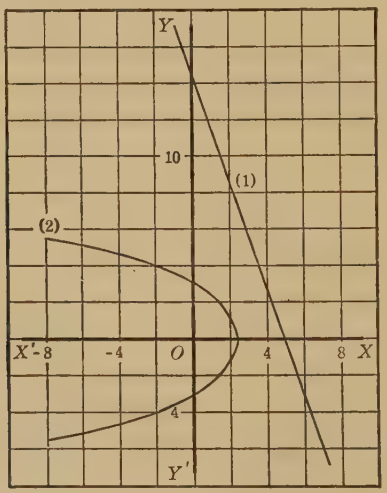
NOTE. If the straight line in the adjacent figure were moved to the right in such a way that it always remained parallel to its present

position, the points A and B would approach each other and finally coincide. The line would then be tangent to the parabola.

Were the straight line moved still farther, it would neither touch nor intersect the parabola and there would be no graphical solution (see page 185).

2. Solve graphically $\begin{cases} 3x + y = 15, & (1) \\ y^2 + 3x = 7. & (2) \end{cases}$

Solution. The graphs of (1) and (2) are the straight line and the parabola of the adjacent figure. These curves have no



real points of intersection. There are, however, two pairs of imaginary roots, for as will be seen in the next chapters solving (1) and (2) by substitution gives

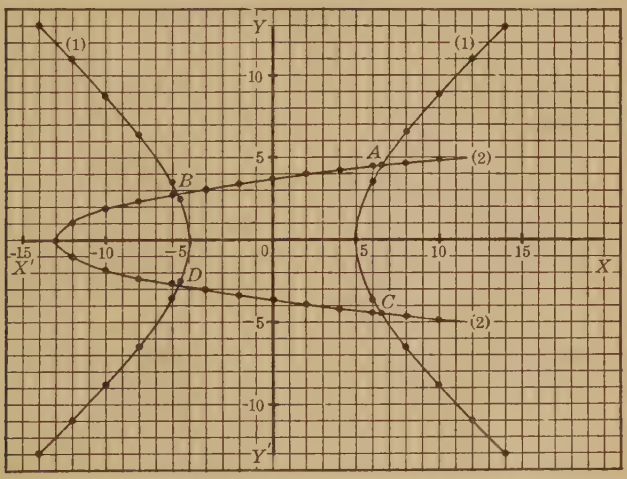
$$A\begin{cases} x = \frac{29}{6} - \frac{1}{6}\sqrt{-31}, \\ y = \frac{1}{2} + \frac{1}{2}\sqrt{-31}. \end{cases} \qquad B\begin{cases} x = \frac{29}{6} + \frac{1}{6}\sqrt{-31}, \\ y = \frac{1}{2} - \frac{1}{2}\sqrt{-31}. \end{cases}$$

The essential point to be emphasized here is that real roots of a simultaneous system correspond to real inter-

sections, and imaginary roots correspond to no intersections of real graphs.

3. Solve graphically $\begin{cases} x^2 - y^2 = 25, & (1) \\ y^2 - x - 12 = 0. & (2) \end{cases}$

Solution. Constructing the graphs of (1) and (2), we obtain the hyperbola and the parabola of the adjacent figure. There



are four sets of roots corresponding to the four points of intersection, which are approximately

$$A\begin{cases} x = 6.6, \\ y = 4.3. \end{cases} \qquad B\begin{cases} x = -5.6, \\ y = 2.5. \end{cases}$$

$$C\begin{cases} x = 6.6, \\ y = -4.3. \end{cases} \qquad D\begin{cases} x = -5.6, \\ y = -2.5. \end{cases}$$

If the two curves had been so chosen as to intersect only twice, their equations would have had only two sets of real roots.

Examples 1, 2, and 3 partially illustrate the truth of the following statement:

If in a system of two equations in two variables one equation is of the mth degree and one of the nth, there are *usually* mn sets of roots (real or imaginary) *and never more than* mn *such sets.*

EXERCISES

If possible solve graphically each of the following systems:

1. $\begin{cases} y^2 = 9x, \\ 2x + y = 6. \end{cases}$

2. $\begin{cases} x^2 + y^2 = 4, \\ y + 2x = 7. \end{cases}$

3. $\begin{cases} xy = 8, \\ 2x + 3y = 10. \end{cases}$

4. $\begin{cases} x^2 + y^2 = 9, \\ x^2 + y^2 = 4. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 16, \\ x + y = 12. \end{cases}$

6. $\begin{cases} x^2 + y^2 = 25, \\ x^2 - y^2 = 16. \end{cases}$

7. $\begin{cases} x^2 + y^2 = 36, \\ x^2 - y^2 = 144. \end{cases}$

8. $\begin{cases} xy = 9, \\ 2x + 3y = 4. \end{cases}$

9. $\begin{cases} x^2 = 9y, \\ 4x^2 + y^2 = 4. \end{cases}$

10. $\begin{cases} x^2 + 3y = 15, \\ x + 3y = 10. \end{cases}$

11. $\begin{cases} x^2 + 3y = 15, \\ x + 3y = 25. \end{cases}$

12. $\begin{cases} x^2 + y = 7, \\ y^2 + x = 4. \end{cases}$

13. $\begin{cases} x - y\sqrt{3} = 0, \\ x^2 = y^3 - 16y. \end{cases}$

106. Graphical presentation of numerical data. A great variety of statistics can be presented graphically in a very striking manner. Business and commercial houses have during the past few years used the method extensively not only to present facts but also to aid in interpreting them and in indicating their meaning.

The following exercises illustrate one of the most striking and important features of the graphical presentation of numerical data.

EXERCISES

1. For the year 1918 personal incomes between $4000 and $15,000 in the United States were grouped as follows:

Annual Income	Number of Persons (thousands)	Annual Income	Number of Persons (thousands)
$4,000 to $5,000	430	$10,000 to $11,000	36
5,000 to 6,000	235	11,000 to 12,000	28
6,000 to 7,000	143	12,000 to 13,000	22
7,000 to 8,000	95	13,000 to 14,000	18
8,000 to 9,000	67	14,000 to 15,000	15
9,000 to 10,000	48		

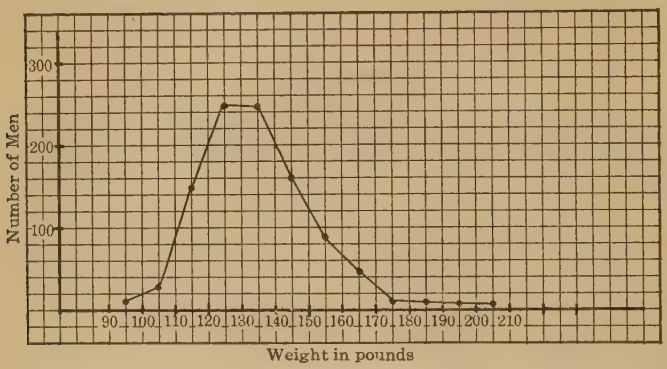
Graph these figures. From this curve estimate the number of persons having an income of $15,000 to $16,000 per year.

Hint. Plot the income on the horizontal axis, and the number of persons on the vertical axis.

2. The weights of 1000 men chosen at random were distributed as follows:

Weight (Lbs.)	Number of Men	Weight (Lbs.)	Number of Men
90 to 100	13	150 to 160	89
100 to 110	28	160 to 170	46
110 to 120	146	170 to 180	18
120 to 130	245	180 to 190	9
130 to 140	242	190 to 200	3
140 to 150	160	200 to 210	1

Plot these figures, using the vertical axis for the number of men in each weight group.



Curves of this general form are very characteristic of certain classes of physical measurement. A few instances are given in the following exercises.

3. Measurements were taken on the lengths of pods of a certain variety of bean. The following figures were obtained from this study.

Length of Pod (quarter inches)	Number	Length of Pod (quarter inches)	Number
12	2	18	85
13	7	19	81
14	25	20	56
15	40	21	13
16	63	22	4
17	92	23	1

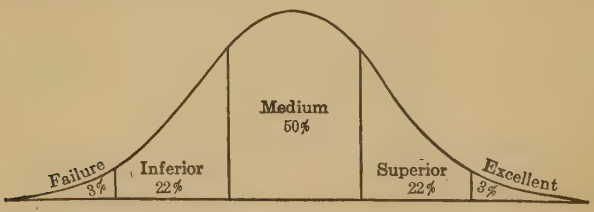
Draw the length distribution curve for this set of figures and compare it with the preceding curve.

4. In a certain college the class marks in English courses were distributed in the following manner:

6. Construct the graph of the equation $y = e^{-x^2}$ between the values $x = -2$ and $x = 2$.

HINT. Let $x = -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0$, etc.

NOTE. It is well established that physical characteristics, such as those illustrated by the graphs of Exercises 2 and 3, obey the law of probability. The graph of Exercise 4 raises the question, Do mental characteristics also obey this law? An interesting aspect of this is given by the fact that an increasing number of high schools and colleges assume that such is the case, and grade, or mark, their students by a system based on the law of probability. Such a system assumes that if one hundred or more students in any subject are examined, the number of students and their degree of mastery of the subject arrange themselves according to the adjacent probability curve. Obviously between very many students the differences in grades attained will be small, between many others they will be moderate, and between only a few will they be great.



In the statistical study of problems which have their origin quite remote from each other, this curve frequently occurs, and occupies a central position in the mathematical theory of statistics.

CHAPTER XV

SYSTEMS SOLVABLE BY QUADRATICS

107. Introduction. The algebraic solution of the system $x^2 + y = 7$ and $y^2 + x = 11$ leads to an equation of the fourth degree. For substituting in the second equation the value of y from the first gives $(7 - x^2)^2 + x = 11$ or $x^4 - 14x^2 + x + 38 = 0$. Similarly, given a pair of equations each of which is a general equation of the second degree such as $ax^2 + by^2 + cxy + dx + ey + f = 0$, we may regard one as a quadratic equation and solve it for x in terms of y and substitute the result in the other. This also gives an equation of the fourth degree which must then be solved for y, and after that is done the corresponding values x can be obtained. The algebraic solution of such systems will not be possible for the student without an advanced course in algebra. It should be noted, however, that by the methods of Chapter XIV one can solve such systems as the above for real roots provided the terms have numeric coefficients. Moreover, many systems which frequently arise in practice can be solved algebraically since the result of substituting from one into the other gives an equation which can be solved by quadratics. Such systems will now be studied.

108. Linear and quadratic systems. Every system of equations in two variables in which one is of the first degree (linear) and the other is of the second degree (quadratic) can be solved by the method of substitution.

EXAMPLE

Solve the system $\begin{cases} 2x^2 - 3xy = 5, & (1) \\ 2x + 3y = 19. & (2) \end{cases}$

Solution. Solving (2), $y = \dfrac{19 - 2x}{3}$. (3)

Substituting $\dfrac{19 - 2x}{3}$ for y in (1),

$$2x^2 - 3x \cdot \frac{(19 - 2x)}{3} = 5. \quad (4)$$

From (4), $4x^2 - 19x = 5$. (5)

Solving (5), $x = 5, -\frac{1}{4}$. (6)

Substituting 5 for x in (3), $y = \dfrac{19 - 10}{3} = 3$. (7)

Substituting $-\frac{1}{4}$ for x in (3), $y = \dfrac{19 + \frac{1}{2}}{3} = 6\frac{1}{2}$. (8)

Hence, for $x = 5, y = 3,$

and for $x = -\frac{1}{4}, y = 6\frac{1}{2}$. (9)

Check. Substituting 5 for x and 3 for y in (1) and (2):

$$50 - 45 = 5, \quad (10)$$

$$10 + 9 = 19. \quad (11)$$

Substituting $x = -\frac{1}{4}$ and $y = 6\frac{1}{2}$ in (1) and (2),

$$\tfrac{1}{8} + \tfrac{39}{8} = 5, \quad (12)$$

$$-\tfrac{1}{2} + \tfrac{39}{2} = 19. \quad (13)$$

The student should note that many pairs of values of x and y can be found which will satisfy either of the two original equations. Hence in checking it is necessary to substitute each set of roots in *both* equations in order to prove that the roots obtained are correct.

It will be found that many systems in two unknowns in which one is of the first degree (linear) and the other is of the third or even a higher degree can be solved by the method of substitution. Thus, for example, the system $x^3 - y^3 = 7$ and $x - y = 3$ can be solved by substitution. So also can the system $x^4 + y^4 = 17$ and $x + y = 3$. Obviously if the result of substitution gives an equation of the third or fourth degree which can be solved by factoring, the algebraic solution can be completely carried out.

Occasionally when neither equation is linear one can be solved for x in terms of y and the solution completed by substitution. In the system $\left\{\begin{array}{l} 2x^2 + 5xy + 3y^2 = 0 \\ x^2 + xy = 16 \end{array}\right\}$ the first equation can be solved for x in terms of y and then substitution in the second can be used to finish the solution.

EXERCISES

Solve the following systems, pair results, and check each set of roots.

1. $x^2 + y^2 = 68$, $x = 4y$.
2. $x^2 + 5y = 29$, $x + y = 7$.
3. $x + y = 9$, $x^2 + y^2 = 41$.
4. $s^2 - t^2 = 5$, $3s + t = 11$.
5. $x^2 + xy = 10$, $x + 2y = 8$.
6. $y^2 - 3yz = 0$, $2y + z = 7$.
7. $2y + 3yz = -2$, $y - 3z = 5$.
8. $2y^2 + 3yz = 8$, $3y + 2z = 7$.
9. $4s^2 - 3t^2 = 24$, $2s - 3t = 12$.
10. $3s^2 - 4st = 80$, $3s + 2t = 8$.
11. $s^2 + 5t^2 = 24$, $4s - 3t = 14$.
12. $s^2 + st + t^2 = 9$, $s + t = 0$.

13. $5y + z^2 = 3yz$,
$y - 2z = 0$.

14. $s^2 - st + t^2 = 84$,
$s + t = 12$.

15. $yz = 60$,
$4y + 3z = 56$.

16. $2y + 5z = 5$,
$yz + 4z = -42$.

17. $x_1 - 10x_2 = 22$,
$x_1x_2 - 6x_2 = -6$.

18. $\frac{s}{t} + \frac{t}{s} = \frac{5}{2}$,
$s + 2t = 8$.

19. $\frac{1}{x} - \frac{1}{z} = 4$.
$3x - 5z = 4$.

20. $\frac{x}{3} - \frac{y}{6} = 8$,
$\frac{9}{x} - \frac{4}{y} = \frac{5}{6}$.

21. $\frac{27}{x} + \frac{y}{3} = 1$,
$\frac{3}{y} - \frac{x}{18} = -1$.

22. $.8x + 3y = \frac{3}{2}$,
$\frac{y}{x} = \frac{7}{5}$.

23. $\frac{s+2}{t+4} = \frac{1}{2}$,
$\frac{t}{s-1} = \frac{s}{t-8}$.

24. $x + 10xy = 19.5$,
$x + 100y = 18$.

25. $\frac{t+1}{s-4} = \frac{s+1}{s-3}$,
$\frac{10}{s} + \frac{4}{t} = 4$.

26. $\frac{162}{y} + \frac{21}{x} = 100$,
$2x - 6y = 5$.

27. $s^2 + t^2 - 2ts = 100$,
$s - 2t = 13$.

28. $x^3 - y^3 = 56$,
$x - y = 2$.

29. $v^2 + u^2 + 4u - 6v = 23$,
$2v - 5u = -14$.

30. $\frac{x^2}{9} + \frac{y^2}{16} = 2$,
$5x + 3y = 3$.

31. $yz + 2z^2 = 2a^2 - 2ac$,
$y - 2z = 2a + 2c$.

32. $xy = a^2$,
$x - y = a$.

33. $\frac{y}{a-z} + \frac{a-z}{y} = \frac{5}{2}$,
$y + z = c$.

34. $yz = a(y - z)$,
$a(y - z) = c(y + z)$.

109. Homogeneous equations. If both the equations of a system are quadratic, an attempt to solve it by substitution gives an equation of the fourth degree. In most cases such an equation cannot be solved by factoring and its algebraic solution by any other method is beyond the student at present. With certain types of systems, however, which occur more or less frequently, we can employ special devices and avoid the solution of an equation of higher degree than a quadratic. Among these systems are the so-called " homogeneous " systems.

An equation is *homogeneous* if, on being written so that one member is zero, the terms in the other member are of the same *degree* with respect to the two variables.

Thus $5xy - x^2 = y^2$ and $x^2 + 8xy + 6y^2 = 0$ are homogeneous equations of the second degree. A homogeneous equation of the third degree is $3x^3 - x^2y + 4xy^2 = 8y^3$.

The system $\left\{\begin{array}{l} 5xy - x^2 = y^2 \\ 2y^2 - 5xy - 2y^2 = 0 \end{array}\right\}$ is a homogeneous system.

Systems like $\left\{\begin{array}{l} x^2 - xy + y^2 = 7 \\ 2x^2 + 3y^2 - xy = 4 \end{array}\right\}$ are often called homogeneous. As will be seen, the method of solving such systems is about the same as the method of solving a homogeneous system. Hence they are classed with homogeneous systems.

110. Systems having both equations quadratic. Occasionally when both equations are quadratic, the terms which occur in the two are so related that the elimination of terms involving both variables or of the constant terms can be performed. The system in the first example below is of such a type.

EXAMPLE

1. Solve the system $\begin{cases} 4x^2 - 3xy = 18, & (1) \\ x^2 + xy = 15. & (2) \end{cases}$

Solution. First eliminate xy by addition (section 52).

(2)·3, $\quad 3x^2 + 3xy = 45 \quad (3)$

(1)+(3), $\quad 7x^2 = 63. \quad (4)$

From (4) $\quad x = \pm 3.$

Substituting 3 for x in (2), $9 + 3y = 15$, whence $y = 2$.

Substituting -3 for x in (2), $9 - 3y = 15$, whence $y = -2$.

Therefore $\quad x = 3, -3,$

and $\quad y = 2, -2.$

These values can be checked as usual.

The method of solving a system in which every term is of the second degree except the constant term is as follows:

2. Solve the system $\begin{cases} x^2 - xy + y^2 = 7, & (1) \\ y^2 - 2xy = -3. & (2) \end{cases}$

Solution. First we eliminate the constant terms, 7 and -3, in order to obtain a homogeneous equation.

(1)·3, $\quad 3x^2 - 3xy + 3y^2 = 21. \quad (3)$

(2)·7, $\quad 7y^2 - 14xy = -21. \quad (4)$

(3)+(4), $\quad 3x^2 - 17xy + 10y^2 = 0. \quad (5)$

Solving (5) for x in terms of y, $\quad x = 5y. \quad (6)$

$$x = \frac{2y}{3}. \quad (7)$$

Substituting $x = 5y$ in (2),

$$y^2 - 10y^2 = -3. \quad (8)$$

Solving (8), $\quad y = \pm\sqrt{\tfrac{1}{3}} = \pm \tfrac{1}{3}\sqrt{3}. \quad (9)$

Substituting from (9) in (6), $x = \pm \frac{5}{3}\sqrt{3}$. (10)

Substituting $\frac{2y}{3}$ for x in (2),

$$y^2 - \frac{4y^2}{3} = -3. \quad (11)$$

Solving (11), $y = \pm 3$. (12)

Substituting from (12) in (7), $x = \pm 2$. (13)

When $x =$	2	-2	$\frac{5}{3}\sqrt{3}$	$-\frac{5}{3}\sqrt{3}$
Then $y =$	3	-3	$\frac{1}{3}\sqrt{3}$	$-\frac{1}{3}\sqrt{3}$

EXERCISES

Solve, pair results, and check:

1. $x^2 - y^2 = 5$,
   $2x^2 + y^2 = 22$.

2. $3x^2 - 2xy = 8$,
   $2x^2 + 3xy = 14$.

3. $x^2 - xy + y^2 = 7$,
   $x^2 + xy - y^2 = 1$.

4. $3xy - x^2 = 8$,
   $xy + 3y^2 = 20$.

5. $2t^2 + st = 5$,
   $st + s^2 = 12$.

6. $2s^2 + t^2 = 41$,
   $s^2 - 4t^2 = -20$.

7. $2s^2 + 3t^2 + 2 = 5$,
   $5t^2 - 3s^2 - 2 = 3$.

8. $x^2 + 2y^2 = 33$,
   $2x^2 - 3xy + y^2 = 24$.

9. $x^2 - 3xy + 2y^2 = 0$,
   $3x^2 - 2xy + y^2 = 18$.

HINT. Solve the first equation for x in terms of y and substitute in the second.

10. $xy - y^2 = 0$,
    $x^2 - 2xy = 1$.

11. $x^2 + y^2 = 125$,
    $3x^2 + 2y^2 = 7xy$.

12. $x^2 + xy = 77$,
    $xy + y^2 = 44$.

HINT. Add the two equations and then obtain $x + y = \pm 11$, etc.

13. $x^2 - xy = 24$,
    $3xy - 4y^2 = -40$.

HINT. Subtract the second equation from the first, etc.

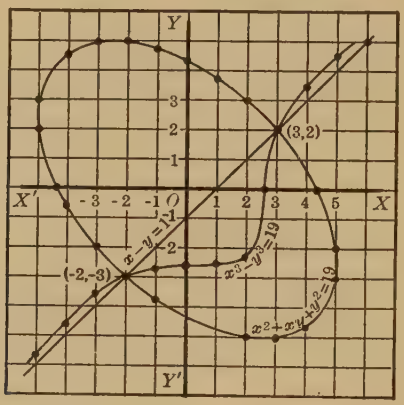
14. $x^2 + xy + y^2 = 52,$
$x^2 - xy + y^2 = 84.$

15. $s^2 + st = 2a^2,$
$t^2 + st = 2a^2.$

16. $s^2 + st = c,$
$t^2 + st = n.$

17. $xy - y^2 = 2ac,$
$x^2 - xy = a^2 + c^2.$

111. Equivalent systems. Equivalent systems of equations are systems which have the same set or sets of roots.



If the three systems

$A.$ $\begin{cases} x^3 - y^3 = 19, & (1) \\ x - y = 1. & (2) \end{cases}$ $B.$ $\begin{cases} x^2 + xy + y^2 = 19, & (3) \\ x - y = 1. & (2) \end{cases}$

and $C.$ $\begin{cases} x^3 - y^3 = 19, & (1) \\ x^2 + xy + y^2 = 19. & (3) \end{cases}$

are solved, the values $\begin{cases} x = 3, -2 \\ y = 2, -3 \end{cases}$ are obtained for each.

Hence systems A, B, and C are equivalent.

The adjacent graphs of the three equations in the systems A, B, and C illustrate the equivalence of the three systems as well as the striking difference between them. In

the graph the two common points of intersection are (3, 2) and (− 2, − 3). This shows that systems A, B, and C have the same sets of roots and hence are equivalent.

In this connection the student should remember that the equivalence of several systems, such as A, B, and C, depends solely on the fact that these systems have the same sets of roots. The form of the individual curves which go to make up the different systems does not in any degree determine whether the systems are equivalent. From this it follows that an unlimited number of equivalent systems can be set up with the roots, (3, 2) and (−2, −3), just as an unlimited number of curves can be drawn through the points (3, 2) and (−2, −3).

112. Symmetric systems. A system of equations in x and y is symmetric if the system is not altered by substituting x for y and y for x.

Thus $x + y = 5$ and $x^2 + xy + y^2 = 20$ form a symmetric system, but $x + y = 5$ and $x^2 + xy - y^2 = 20$ do not.

Certain symmetric systems or systems which are nearly so can be easily solved by the method of substitution. This is true of the following types:

$$\begin{cases} xy = 12, \\ x \pm y = 3. \end{cases} \qquad \begin{cases} x \pm 3y = 7, \\ x^2 + 9y^2 = 37. \end{cases} \qquad \begin{cases} x \pm y = 5, \\ x^2 + y^2 = 25. \end{cases}$$

A few other systems which are symmetric or nearly so are more easily solved by certain special methods. The following list contains typical systems, and the methods applicable are outlined in Exercises 1, 10, 12, which follow.

It should be noted that many systems can be solved by substitution even though neither equation of the system is linear. This is possible with the system $xy = 6$ and

$x^2 + y^2 = 25$. The solution of this system and of many somewhat similar systems can be more easily carried out, however, by the methods which follow.

EXERCISES

1. Solve $\begin{cases} x^2 + y^2 = 45, & (1) \\ xy = 18. & (2) \end{cases}$

HINT. Here the second equation could be solved for x in terms of y and the solution completed by substitution. A simpler method is to combine them in such a way as to obtain numeric values for $x + y$ and $x - y$ as follows:

(2) · 2,	$2xy = 36.$	(3)
(1) + (3),	$x^2 + 2xy + y^2 = 81.$	(4)
From (4),	$x + y = \pm 9.$	(5)
(1) − (3),	$x^2 - 2xy + y^2 = 9.$	(6)
From (6),	$x - y = \pm 3.$	(7)

(5) and (7) combined give the four linear systems:

$A.$ $\begin{cases} x + y = 9, & (8) \\ x - y = 3. & (9) \end{cases}$ $C.$ $\begin{cases} x + y = -9, & (12) \\ x - y = 3. & (13) \end{cases}$

$B.$ $\begin{cases} x + y = 9, & (10) \\ x - y = -3. & (11) \end{cases}$ $D.$ $\begin{cases} x + y = -9, & (14) \\ x - y = -3. & (15) \end{cases}$

These four systems are equivalent to the original one.

The solution of these systems is left to the student.

The pairs of roots found for the four systems A, B, C, D will check in the original system.

2. $x^2 + y^2 = 65,$
$xy = 28.$

3. $4x^2 + y^2 = 104,$
$xy = 10.$

4. $x^2 + xy + y^2 = 19,$
$xy = 6.$

5. $x^2 + 9y^2 = 244,$
$xy = 40.$

6. $x^2 - xy + y^2 = 21,$
$xy = 20.$

7. $x^2 - xy + y^2 = 84,$
$x^2 + xy + y^2 = 244.$

8. $x^2 + xy + 4y^2 = 64,$
$xy = 12.$

9. $4x^2 + 9y^2 = 72,$
$xy = 6.$

10. $\frac{1}{x^2} + \frac{1}{y^2} = 5,$
$\frac{1}{xy} = 2.$

HINT. To clear of fractions would merely increase the difficulty of solution. We can combine the equations more effectively as follows:

$$\frac{2}{xy} = 4.$$

Then $$\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = 9.$$

Or $$\frac{1}{x} + \frac{1}{y} = \pm 3.$$

Similarly $$\frac{1}{x} - \frac{1}{y} = \pm 1.$$

Here as in Exercise 1 we have four systems equivalent to the original one. These four may be solved for $\frac{1}{x}$ and $\frac{1}{y}$ and thus finally the four sets of values of x and y obtained.

11. $\frac{1}{x^2} + \frac{1}{y^2} = 13,$
$\frac{1}{xy} = 6.$

12. $\frac{1}{x^2} + \frac{1}{y^2} = 25,$
$\frac{1}{x} - \frac{1}{y} = 1.$

HINT. $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = 1.$

Then $\frac{2}{xy} = 24,$ etc.

13. $\frac{1}{x^2} + \frac{1}{y^2} = 41,$
$\frac{1}{x} + \frac{1}{y} = 9.$

14. $\frac{1}{9x^2} + \frac{1}{4y^2} = 20,$
$\frac{1}{3x} + \frac{1}{2y} = -2.$

15. $\frac{1}{x} + \frac{1}{y} = \frac{1}{6},$
$\frac{1}{xy} + \frac{1}{18} = 0.$

16. $\frac{1}{x} + \frac{1}{y} = a + b,$
$\frac{1}{xy} = ab.$

17. $\frac{1}{x^2} + \frac{1}{y^2} = 5a^2,$
$xy + \frac{1}{2a^2} = 0.$

113. Use of division in systems of equations. Sometimes an equation simpler than one of a given system can be derived by dividing the left and right members of one equation by the corresponding members of the other. The system composed of this derived equation and the simpler of the two given ones is equivalent to the original system and may be more easily solved.

EXERCISES

Solve, using division where possible, pair results, and check each set of real roots:

1. $4x^2 - y^2 = 16,$
$2x - y = 8.$

HINT. Division gives the equivalent system.

$2x + y = 2,$
$2x - y = 8,$ etc.

2. $x^2 - y^2 = 16,$
$x - y = 8.$

3. $25x^2 - 4y^2 = 0,$
$5x + 2y = 2.$

4. $x^3 + y^3 = 488,$
$x + y = 2.$

5. $x^3 - y^3 = 468,$
$x - y = 12.$

6. $u^3 + v^3 = 296,$
$u^2 - uv + v^2 = 148.$

7. $xy + y^2 = 12,$
$x^2 + xy = 24.$

HINT. $\frac{y(x+y)}{x(x+y)} = \frac{12}{24},$
or $2y = x.$

8. $8x^3 + y^3 = 16,$
$2x + y = 4.$

9. $s^3 - 8t^3 = 35,$
$s - 2t = 5.$

10. $\frac{1}{x} + \frac{1}{y} = 30,$
$\frac{1}{y^2} - \frac{1}{x^2} = 300.$

11. $s^2 - st = 120,$
$t^2 - st = 24.$

12. $\frac{1}{u^3} - \frac{1}{v^3} = 37,$
$\frac{1}{u} - \frac{1}{v} = 1.$

13. $x^4 + x^2y^2 + y^4 = 21,$
$x^2 + xy + y^2 = 7.$

14. $x^2 - y^2 = ab,$
$x + y = a.$

15. $x^3 + y^3 = a^3 - 8,$
$x + y = a - 2.$

MISCELLANEOUS EXERCISES

Solve by any method and pair results. If any system cannot be solved by methods previously given, solve it graphically.

1. $x + y = 4,$
$xy = 3.$

2. $x^2 - xy = -15,$
$2x - y = 14.$

3. $x^2 + xy + y^2 = 7,$
$2x + 3y = 0.$

4. $x^2 + y = 38,$
$x + 5y = 4.$

5. $s^2 - t^2 = 24,$
$s + 2t = 3.$

6. $x^2 + y^2 = 100,$
$xy = 48.$

7. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 5,$
$x - y = 5.$

8. $u^3 + v^3 = 35,$
$u + v = 5.$

9. $\sqrt[3]{x} - \sqrt[3]{y} = 1,$
$x - y = 19.$

10. $yz + z = 20,$
$yz + y = 18.$

11. $x^3 - y^3 = 152,$
$x^2 + xy + y^2 = 19.$

12. $h^2 - hk = 160,$
$hk - k^2 = 96.$

13. $y - xy = 72,$
$3x + xy = -50.$

14. $s^2 + t^2 = 100,$
$s^2 - t^2 = 28.$

15. $x_1^2 + x_1x_2 = 200,$
$x_1x_2 + x_2^2 = -100.$

16. $x + y = 1.7,$
$xy = .52.$

17. $.1s + .1t = .015,$
$.2s^2 + .2t^2 = .0025.$

18. $x + .2y = .94,$
$.2x^2 - xy = -.742.$

19. $x + y = 2a,$
$xy = a^2 - b^2.$

20. $x^2 + xy + y^2 = 3a^2,$
$x^2 + xy = 2a^2.$

21. $m^{\frac{1}{2}} + n^{\frac{1}{2}} = 13,$
$m^{\frac{1}{2}}n^{\frac{1}{2}} = 40.$

22. $4u^2 - v^2 = 84,$
$u - 2v = 13.$

23. $u^2 - uv = 104,$
$uv + v^2 = -15.$

24. $\frac{1}{x^2} - \frac{1}{y^2} = 9,$
$\frac{1}{x} - \frac{1}{y} = 9.$

25. $\frac{1}{x^2} - \frac{1}{y^2} = 19,$
$\frac{1}{xy} = 90.$

26. $\frac{1}{x^3} - \frac{1}{y^3} = 152,$
$\frac{1}{x} - \frac{1}{y} = 8.$

27. $x^{-2} - y^{-2} = 20,$
$x^{-1} - y^{-1} = 2.$

28. $\frac{y - x}{xy + 1} = \frac{3}{19}.$
$\frac{x + y}{xy - 1} = \frac{9}{17}.$

29. $2u^2 + v = 3uv - 12,$
$2u - v = 0.$

30. $v + uv = -40,$
$u + uv = -28.$

31. $xy^2 = 64,$
$x^2y = 8.$

32. $3x^2 + 5xy - 2y^2 = 0,$
$x^2 + 2xy = 175.$

33. $x^3 + y^3 = 189,$
$x^2 - xy + y^2 = 21.$

34. $2x^2 - xy - y^2 = 0,$
$9x^2 - 9y^2 = 5.$

35. $x^3 + y^3 = 126,$
$x^2y + xy^2 = 30.$

PROBLEMS

(Reject all results which do not satisfy the conditions of the problem.)

1. Find two numbers whose difference is 5 and whose product is 14.

2. Find two numbers whose sum is 30 and the difference of whose squares is 300.

3. Find two numbers whose product plus their difference is 28 and whose quotient is 5.

4. The area of a right triangle is 120 and its hypotenuse is 26. Find the other two sides.

5. It requires 52 rods of fence to inclose a rectangular lot whose area is one acre. Find the dimensions of the lot.

6. The area of a field containing 9.6 acres is doubled by adding 16 rods to the length and the breadth. Find the dimensions.

7. The area of a rectangular field is 30 acres and one diagonal is 100 rods. Find the dimensions.

8. If a number containing two digits is divided by the product of the digits, the quotient is 2. If the number formed by reversing the digits is divided by their sum, the quotient is 7. Find the number.

9. Two numbers are formed of the same two digits in reverse order. The sum of their squares is 2340. The smaller divided by the larger is $\frac{4}{7}$. Find the numbers.

10. The difference in volume of two cubes is 386 cubic inches and the difference of their edges is 2 inches. Find the edges.

11. A walks 14 miles in $\frac{1}{2}$ hour less time than B. The latter walks $\frac{1}{2}$ mile per hour slower than A. Find the speed of each.

12. The value of a fraction is decreased $\frac{7}{40}$ when 1 is added to the numerator and 3 to the denominator and it is increased $\frac{1}{5}$ if 1 is subtracted from the numerator and 2 from the denominator. Find the fraction.

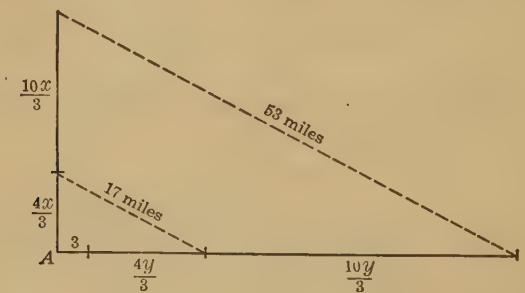
13. If the usual speed of a motor bus is increased 5 miles per hour, 16 minutes will be saved in making a certain run. Twenty-four minutes more will be required on the same run if the rate is decreased 5 miles per hour. Find the length of the run and the usual speed of the bus.

14. The sum of the squares of the digits of a two-digit number is 9 more than the number. Twice the product of the digits equals the number. Find the number.

15. A certain sum at simple interest amounts to \$834 in one year. If the rate were $1\frac{3}{4}$ per cent greater and the sum were \$100 more, the amount would be \$954. Find the sum and the rate.

16. A grocer spent six dollars for oranges. Had they cost four cents a dozen less he would have received five dozen more for the same sum. How many dozen did he buy and what was the price per dozen?

17. A bicyclist leaves A and travels north. At the same time a second bicyclist leaves a point 3 miles east of A and travels east. One and one-third hours after starting, the shortest distance between them is 17 miles. Three and one-third hours later it is 53 miles. Find the speed of each.



18. Divide 365 into two parts so that they will be squares of consecutive integers.

19. The sum of the radii of two circles is 36 inches and the difference of their areas is $144\,\pi$ square inches. Find the radii.

20. The sum of one number and the reciprocal of another equals 10.25. The sum of the second and the reciprocal of the first equals 4.1. Find the two numbers.

21. A starts out from P to Q at the same time B leaves Q for P. When they meet, A has gone 40 miles more than B. A then finishes the journey to Q in 2 hours and B the journey to P in 8 hours. Find the speeds of A and B and the distance from P to Q.

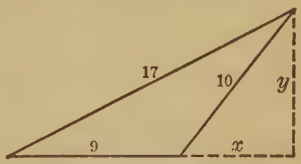
22. A leaves P for Q at the same time that B leaves Q on his way to P. From the time the two meet, it requires $6\frac{2}{3}$ hours for A to reach Q and 15 hours for B to reach P. Find the speed of each if the distance from P to Q is 300 miles.

GEOMETRICAL PROBLEMS

1. The sides of a triangle are 9, 10, and 17. Find the altitude on the side 9.

HINT. From the adjacent figure

$$x^2 + y^2 = 100$$
$$(x + 9)^2 + y^2 = 289$$



2. The sides of a triangle are 7, 15, 20. Find the altitude on the side 7 and the area of the triangle.

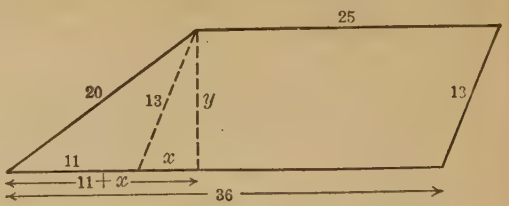
3. What property of a triangle can be used to find the other two altitudes in Exercises 1 and 2? Find the other two altitudes for the triangle in Exercise 2 by this method.

4. The sides of a triangle are 18, 24, and 30. Find the shortest altitude of the triangle.

5. The sides of a triangle are 10, 17, and 21. Find the altitude on the greatest side.

6. The sides of a triangle are 25, 33, and 52. Find the altitude on the side 33.

7. The parallel sides of a trapezoid are 25 and 36 respectively. The non-parallel sides are respectively 13 and 20. Find the altitude of the trapezoid.



HINTS.
$$x^2 + y^2 = 169,$$
$$(11 + x)^2 + y^2 = 400.$$

8. The two parallel sides of a trapezoid are 30 and 51 respectively. The other two sides are 10 and 17 respectively. Find the altitude of the trapezoid and its area.

9. The sides of a trapezoid are 20, 30, 37, and 81 respectively. The sides 30 and 81 are parallel. Find the altitude of the trapezoid and its area.

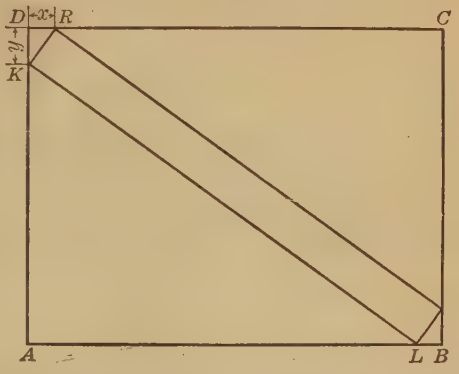
10. The sides of a trapezoid are 14, 18, 40, and 48. The sides 18 and 48 are the bases. Find the altitude and the area of the trapezoid.

11. The sides of a trapezoid are 45, 53, 90, and x. The latter is perpendicular to the sides 45 and 90. Find x and the area of the trapezoid.

12. The hypotenuse of a right triangle is 125 and its area is 2574. Find the other two sides.

13. The area of a right triangle is 96. The difference between the sum of the smaller sides and the hypotenuse is 8. Find the three sides.

14. A rectangular tank is 8 feet 6 inches long and 6 feet 8 inches wide. A board 10 inches wide is laid diagonally on the floor. What two equations must be solved to determine the length of the longest board that can be thus laid?



HINT. Let $DR = x$ and $DK = y$. The triangle DKR is similar to the triangle AKL. If the student desires to find the length of the board, he must solve the system obtained graphically. This is because an attempt to use the other methods presented in this chapter leads to an equation of the fourth degree which cannot be solved by factoring or any simple method. If the student studies advanced algebra, he will learn to solve such equations readily by Horner's method.

CHAPTER XVI

THE BINOMIAL THEOREM

114. Powers of binomials. The following identities are easily obtained by actual multiplication:

$$(a + b)^2 = a^2 + 2ab + b^2 \quad (1)$$

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 \quad (2)$$

$$(a + b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4 \quad (3)$$

$$(a + b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5 \quad (4)$$

If $a + b$ is replaced by $a - b$, the even-numbered terms in each of the preceding expressions will then be negative and the odd-numbered terms will be positive.

115. The expansion of $(a + b)^n$. The form of the expansion for the case where n is an integer is as follows:

The first term is a^n *and the last is* b^n.

The second term is $+ na^{n-1}b$.

The exponents of a *decrease by* 1 *in each term after the first.*

The exponents of b *increase by* 1 *in each term after the second.*

The product of the coefficient in any term and the exponent of a *in that term, divided by the exponent of* b *increased by* 1, *gives the coefficient of the next term.*

The sign of each term is $+$ *if* a *and* b *are positive; the sign of each even-numbered term is* $-$ *if* b *alone is negative.*

According to the above statement we have

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \cdots + b^n.$$

This expresses in symbols the law known as the *binomial theorem*. The theorem holds for all positive values of n and also, with certain limitations, for negative values.

NOTE. The coefficients of the various terms in the binomial expansion are displayed in a most elegant form as follows:

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
. . . . . .

In this arrangement each row may be derived from the one above it by observing that each number is equal to the sum of the two numbers, one to the right and the other to the left of it, in the line above. Thus $4 = 1 + 3$, $6 = 3 + 3$, etc. The next line is 1 5 10 10 5 1. The successive lines of this table give the coefficients for the expansions of $(a + b)^n$ for the various values of n. The numbers in the lowest line of the triangle are seen to be the coefficients when $n = 4$; the next line would give those for $n = 5$. This array is known as Pascal's triangle and was published in 1665. It was probably known to Tartaglia nearly a hundred years before its discovery by Pascal.

EXERCISES

Expand by the rule:

1. $(a + b)^6$.
2. $(a - c)^7$.
3. $(a + 1)^6$.
4. $(a - 1)^7$.
5. $(a + 2)^6$.
6. $(a + 3)^5$.
7. $(3 + a)^5$.
8. $(4 - a)^6$.
9. $(5 - a)^8$.

Obtain the first four terms of:

10. $(a + b)^{10}$. 12. $(a + b)^{25}$. 14. $(a - 2)^{18}$.

11. $(a - b)^{15}$. 13. $(a + 1)^{20}$. 15. $(a - 5)^{30}$.

Expand:

16. $(a^2 + 3\,c)^5$.

HINTS. To avoid errors in the exponents write

$(a^2)^5 + 5(a^2)^4(3\,c) + 10(a^2)^3(3\,c)^2 + 10(a^2)^2(3\,c)^3 + 5(a^2)(3\,c)^4 + (3\,c)^5$

Then in the spaces left for them put in the coefficients of the terms according to the binomial theorem. Finally simplify each term.

17. $(a^3 + c)^6$. 18. $(a^2 + 3)^7$. 19. $(a^2 - 2\,c^2)^5$.

20. $\left(\frac{a}{c} + c^2\right)^6$. 21. $\left(\frac{a^2}{c} - \frac{c^3}{a}\right)^4$.

Obtain in simplest form the first four terms of:

22. $(a + 5\,c)^{10}$. 23. $\left(\frac{a}{c} - c^2\right)^{12}$. 24. $(a^2 - 2\,c)^{20}$.

25. $\left(\frac{a}{c} + \frac{c}{a}\right)^{10}$. 28. $\left(\frac{a^2}{x} - \frac{x^2}{2\,a}\right)^{12}$.

26. $\left(\frac{a^2}{c} + c^2\right)^{20}$. 29. $\left(\frac{6\,x^3}{a} + \frac{2\,a^2}{9\,x}\right)^5$.

27. $\left(\frac{a}{x^2} - \frac{2\,x}{a^2}\right)^{14}$. 30. $\left(\frac{\sqrt{a}}{x} - \frac{\sqrt{x}}{a}\right)^{10}$.

31. Write the first six terms of the expansion of $(a + b)^n$, and evaluate it for $n = 1$, $n = 2$, $n = 3$, $n = 4$. How does the number of terms compare with n? What is the value of each coefficient after the $(n + 1)$th? Why does not the expansion extend to more than five terms when $n = 4$?

32. $\left(1 + \frac{1}{n}\right)^n$.

33. $(4\,m^{\frac{3}{2}} - m^{\frac{1}{2}}a^{\frac{1}{3}})^6$.

Compute the following to two decimal places:

34. $(2.1)^8$.

HINT. $(2.1)^8 = (2 + .1)^8$, etc.

35. $(4.2)^{10}$.

36. $(1.04)^7$.

37. $(.97)^{10}$.

HINT. $(.97)^{10} = (1 - .03)^{10}$.

38. $(5.8)^{12}$.

116. Expansion in series. The expansion of $(a+b)^n$ can be proved for small integral values of n by direct multiplication. The theorem holds, however, for any integral value of n. The proof of this fact is not difficult, but it requires a knowledge of mathematical induction which usually comes in an advanced course. For positive integral values of n the number of terms is always $n + 1$. For fractional and negative values of n this is not the case. The reason is that the factors in the coefficients n, $n - 1$, $n - 2$, etc., cannot become zero for fractional or negative values of n, and hence the expansion has no final term and it becomes an infinite series. The expansion is still valid if a is greater than b and is useful for two purposes, the expansion in series and the extraction of roots.

For example

$$(a + b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \tfrac{1}{2}\,a^{-\frac{1}{2}}b - \tfrac{1}{8}\cdot a^{-\frac{3}{2}}b^2 + \tfrac{1}{16}\cdot a^{-\frac{5}{2}}b^3 + \cdots.$$

By giving n the values $\frac{1}{2}$ or $\frac{1}{3}$, one can compute the square root or the cube root of a number to any required degree of accuracy.

In such computations it is desirable to let the number which corresponds to a in the binomial exceed the one corresponding to b by as much as possible and at the same time to have a an integer.

As will be seen, many simple expansions with fractional and negative exponents can be easily verified. That the binomial theorem holds for all fractional and negative values of n when a is greater than b has been proved.

NOTE. The process of extracting the square root and even the cube root by means of the binominal expansion was familiar to the Hindus more than a thousand years ago. The German Stifel (1486–1567) stated the binomial theorem for all powers up to the seventeenth, and also extracted roots of numbers by this method.

EXERCISES

Find the first five terms of:

1. $(1 + x)^{-1}$.

2. Obtain the result of Exercise 1 by actual division.

HINT. $(1+x)^{-1} = \dfrac{1}{1+x}$, etc.

3. $(1 + x)^{-2}$.

4. Obtain the result of Exercise 3 by division.

5. $(28)^{\frac{1}{2}}$.

Solution.

$$\begin{aligned} 28^{\frac{1}{2}} &= (25 + 3)^{\frac{1}{2}} \\ &= (25)^{\frac{1}{2}} + \tfrac{1}{2}(25)^{-\frac{1}{2}}(3) - \tfrac{1}{8}(25)^{-\frac{3}{2}}(3)^2 + \tfrac{1}{16}(25)^{-\frac{5}{2}}(3)^3 \cdots \\ &= 5 + .3 - .009 + .00054 - \cdots = 5.29154. \end{aligned}$$

It is proved in more advanced books that when the terms of an infinite series are alternately plus and minus, and each term is numerically less than the preceding one, the value of the entire sum from a given term on cannot exceed that term. This fact renders these so-called "alternating series" especially convenient for computation, since a definite limit of error is known at each stage of the computation. In this example the error cannot exceed .00054.

6. Check the result of Exercise 5 by extracting the square root by the method of section 64.

7. Show that $(65)^{\frac{1}{3}} = 4.0207$.

Find to three decimals by the binomial theorem:

8. $(26)^{\frac{1}{2}}$. 10. $(38)^{\frac{1}{2}}$.

9. $(27)^{\frac{1}{3}}$. 11. $(83)^{\frac{1}{2}}$.

12. $(62)^{\frac{1}{2}}$.

HINT. $(62)^{\frac{1}{2}} = (64 - 2)^{\frac{1}{2}}$.
Here $(64 - 2)^{\frac{1}{2}}$ yields more accurate results with fewer terms than $(49 + 13)^{\frac{1}{2}}$.

13. $(28)^{\frac{1}{3}}$. 14. $(61)^{\frac{1}{3}}$. 15. $(29)^{\frac{1}{3}}$.

16. $(520)^{\frac{1}{3}}$. 17. $(17)^{-\frac{1}{2}}$.

18. Check the result of Exercise 17 by finding the value of $\frac{1}{\sqrt{17}}$ by the ordinary method of square root. Here $\frac{1}{\sqrt{17}} = \frac{\sqrt{17}}{17}$, etc.

Expand to five terms:

19. $(1 + x)^{\frac{1}{2}}$. 20. $(2 - x)^{\frac{1}{2}}$. 21. $(3 + x)^{\frac{1}{3}}$.

22. $(2 + x)^{-\frac{1}{2}}$. 24. $(1 - x)^{\frac{2}{3}}$.

23. $(8 - x)^{-\frac{1}{3}}$. 25. $(1 + x)^{-\frac{2}{3}}$.

117. The factorial notation. The notation $5!$ or $\underline{|5}$ signifies $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, or 120. Also $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.
In general, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n - 2)(n - 1)n$.
The symbol $n!$ or $\underline{|n}$ is read "factorial n."

In the factorial notation the denominators of the fourth and fifth terms of the expansion of $(a + b)^n$ become $3!$ and $4!$ respectively (see formula, p. 229).

EXERCISES

Evaluate:

1. $7!$.
2. $5! \cdot 3!$.
3. $6! \cdot 4!$.
4. $7! \div 4!$.
5. $5! - (4!)^2$.
6. $8! \div 6!$.
7. $n! \div (n-1)!$.
8. $n! \div (n-2)!$.
9. $(2n)! \div (2n-3)!$.
10. $(n+3)! \div (n-1)!$.

11. Evaluate $\dfrac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}$ when $n = 6, r = 4$; $n = 12, r = 9$; $n = 20, r = 14$; $n = 18, r = 12$.

118. The rth term of $(a + b)^n$. According to the binomial theorem the fifth term of the expansion on page 229 is

$$+\frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{4!}.$$

If we note carefully this term and the directions on page 228, we can write down, from the considerations that follow, any required term without writing other terms of the expansion.

The *denominator of the coefficient* of the fifth term is $4!$. From the law of formation the denominator of the sixth term would be $5!$, of the seventh term $6!$, etc. Consequently in the rth term the denominator of the coefficient would be $(r-1)!$.

The *numerator of the coefficient* of the fifth term contains the product of the four factors $n(n-1)(n-2)(n-3)$.

The numerator of the sixth term would contain these four and the additional factor $n - 4$. Similarly, the last factor in the numerator of the seventh term would be $n - 5$, etc. Hence the last factor in the rth term would be $n - (r - 2)$, and the numerator of the coefficient of the rth term is $(n - 1)(n - 2)(n - 3) \cdots (n - r + 2)$.

The *exponent of* a in the fifth term is $n - 4$, and in the sixth term it would be $n - 5$, etc. Therefore in the rth term the exponent of a is $n - (r - 1)$, or $n - r + 1$.

The *exponent of* b in the fifth term is 4, in the sixth term 5, etc. Therefore in the rth term the exponent of b is $r - 1$.

The *sign* of any term of the expansion (if n is a positive integer) is plus if the binomial is $a + b$. If the binomial is $a - b$, the terms containing the odd powers of b will be negative. In other words the sign in such cases depends upon whether the exponent $r - 1$ is even or odd, being $+$ if $r - 1$ is even, and $-$ if $r - 1$ is odd.

Hence the rth term (r not equal to 1) of $(a + b)^n$ equals

$$(-1)^{r-1} \frac{n(n-1)(n-2)(n-3) \cdots (n-r+2)}{(r-1)!} a^{n-r+1} b^{r-1}. \quad (1)$$

If we wanted the twelfth term, we would in using (1) substitute 12 for r.

EXERCISES

Find the required term:

1. The sixth term of $(a + b)^{12}$.

Solution. Substituting 12 for n and 6 for r in the formula (1) gives

$$+ \frac{12 \cdot 11 \cdot 10 \cdot 9 \cdot 8}{5 \cdot 4 \cdot 3 \cdot 2} \cdot a^7 b^5$$
$$= + 792\, a^7 b^5$$

2. Fifth term of $(a + b)^{10}$.

3. Sixth term of $(a + b)^{13}$.

4. Seventh term of $(a - b)^{11}$.

5. Eighth term of $(a - b)^{10}$.

6. Fourth term of $(a^2 - 2x)^{10}$.

7. Fifth term of $\left(a - \frac{1}{a}\right)^{20}$.

8. Sixth term of $\left(\frac{a}{c} - \frac{c}{a}\right)^{14}$.

9. Eighth term of $(x^3 - x^2)^{16}$.

10. Fourth term of $\left(\sqrt{\frac{a}{x}} - \sqrt{x}\right)^{20}$.

11. Find the middle term of $(x^3 - x^2)^{14}$.

Find the coefficient of:

12. x^4 in $(1 + x)^{12}$.

13. x^{10} in $(x^2 + 1)^{20}$.

14. x^{12} in $(x^3 - 1)^{18}$.

15. x^5 in $(x^3 - x^{-2})^{15}$.

NOTE. The binomial theorem occupies a remarkable place in the history of mathematics. By means of it Napier was led to the discovery of logarithms, and its use was of the greatest assistance to Newton in making his most wonderful mathematical discoveries. But to-day the results of Newton and of Napier are explained without even so much as a mention of the binomial theorem, for simpler methods of obtaining these results have been discovered.

It was Newton who first recognized the truth of the theorem, not only for the case where n is a positive integer, which had long been familiar, but for fractional and negative values as well. He did not give a demonstration of the general validity of the binomial development, and none even passably satisfactory was given until that of Euler (1707–1783). The first entirely satisfactory proof of this difficult theorem was given by the brilliant young Norwegian Abel (1802–1829).

CHAPTER XVII

LOGARITHMS

119. Introduction. Logarithms were invented to shorten the work of extended numeric computations which involve one or more of the operations of multiplication, division, involution, and evolution. Their use has decreased the labor of computing to such an extent that many calculations which would require hours without the use of logarithms can be performed with their aid in a small fraction of that time.

120. Definition of logarithm and base. If we write the equation

$$n = b^l, \tag{1}$$

we express therein the essential relation between a number, n, and its logarithm, l, for a given base, b. In the notation of logarithms this is written

$$\log_b n = l, \tag{2}$$

and it is read " the logarithm of n to the base b equals l."

We can define verbally in one statement both logarithm and base as follows:

The logarithm of a given number is the power to which another number, called the base, must be raised in order to equal the given number.

It is important to realize that equations (1) and (2) are merely two different ways of expressing precisely the same

relations, one the exponential way, the other the logarithmic. Above all it is necessary to keep in mind the fact that a logarithm is an exponent.

Thus in $81 = 3^4$, the given number is 81, the base is 3, and the logarithm is 4; that is, $\log_3 81 = 4$.

121. Systems of logarithms. The base of the common, or Briggs, system of logarithms is 10. Hence a table of common logarithms is really a table of exponents of the number 10. Since the greater portion of these exponents are approximate values of irrational numbers, it follows that computations by means of logarithms give only approximate results. Tables exist, however, in which each logarithm is given to twenty or more decimals; hence practically any desired degree of accuracy can be obtained by using the proper table. The common system is used in numerical work almost exclusively. The table on page 244 is a table of common logarithms carried to four decimal places.

The only other system of logarithms used in computations is called the natural system. It has for its base the irrational number 2.7182+, which is usually denoted by the letter e and is used mainly for theoretical purposes.

It can be proved that the laws given on pages 101–102, governing the use of rational exponents, hold for irrational exponents. In the work on logarithms this fact will be assumed.

ORAL EXERCISES

1. If $16 = 2^x$, $x = ?$ $\log_2 16 = ?$
2. If $625 = 5^x$, $x = ?$ $\log_5 625 = ?$
3. $\text{Log}_4 64 = ?$ $\log_2 32 = ?$ $\log_3 243 = ?$
4. If $10000 = 10^x$, $x = ?$ $\log_{10} 10000 = ?$

5. $\log_{10} 100 = ?$ $\log_{10} 1000 = ?$ $\log_{10} 10 = ?$ $\log_{10} 1 = ?$

6. $4^0 = ?$ $\log_4 1 = ?$ $\log_6 1 = ?$ $\log_n 1 = ?$

	Number	Base	Logarithm		Number	Base	Logarithm
7.	25	5	?	22.	?	10	2
8.	81	3	?	23.	?	10	1
9.	81	9	?	24.	?	10	0
10.	64	2	?	25.	?	10	3
11.	125	5	?	26.	?	10	-1
12.	216	6	?	27.	?	10	-2
13.	343	7	?	28.	?	10	-3
14.	27	?	3	29.	4	?	$\frac{1}{2}$
15.	8	?	3	30.	8	?	$\frac{1}{3}$
16.	128	?	7	31.	27	?	$\frac{3}{2}$
17.	625	?	3	32.	5	25	?
18.	1024	?	5	33.	6	216	?
19.	?	4	4	34.	16	?	$\frac{3}{4}$
20.	?	3	5	35.	81	?	$\frac{3}{4}$
21.	?	5	3	36.	25	?	$\frac{3}{2}$

Read in the notation of logarithms:

37. $73 = 10^{1.863}$.
38. $50 = 10^{1.69}$.
39. $1 = 10^0$.
40. $.1 = 10^{-1}$.
41. $.2 = 10^{-.699}$.
42. $.089 = 10^{-1.05}$.
43. $1650 = 10^{3.2175}$.
44. $165 = 10^{2.2175}$.
45. $16.5 = 10^{1.2175}$.
46. $1.65 = 10^{.2175}$.
47. $.165 = 10^{-1+.2175}$.
48. $.0165 = 10^{-2+.2175}$.
49. $200 = 10^{2.301}$.
50. $20 = 10^{1.301}$.

Read in exponential notation, with the base 10:

51. $\log 3 = .4771.$ 52. $\log 40 = 1.6021.$

53. $\log .0983 = -2 + .9926.$

54. $\log 100^3 = 6.$

Find the numeric value of the following sums:

55. $\log_{10} 10 + \log_{10} 100 + \log_{10} 1000.$

56. $\log_{10} 100 + \log_{10} 1 + \log_{10} .1.$

57. $\log_3 9 + \log_4 16 + \log_5 125.$

58. $\log_7 49 + \log_8 512 + \log_2 4.$

59. $\log_2 8 + \log_3 27 + \log_4 64.$

60. $\log_{36} 6 + \log_{25} 5 + \log_5 25.$

Biographical Note. John Napier. Although many scientists have been honored with titles on account of their discoveries, very few of the titled aristocracy have become distinguished for their mathematical achievements. A notable exception to this rule is found in John Napier, Lord of Merchiston (1550–1617), who devoted most of his life to the problem of simplifying arithmetic operations. Napier was a man of wide intellectual interests and great activity. In connection with the management of his estate he applied himself most seriously to the study of agriculture, and experimented with various kinds of fertilizers in a somewhat scientific manner, in order to find the most effective means of reclaiming soil. He spent several years in theological writing. When the danger of an invasion by Philip of Spain was imminent, he invented several devices of war. Among these were powerful burning mirrors, and a sort of round musket-proof chariot, the motion of which was controlled by those within, and from which guns could be discharged through little portholes.

But by far the most serious activity of Napier's life was the effort to shorten the more tedious arithmetic operations. He invented the first approximation to a computing machine, and also devised a set of rods, often called Napier's bones, which were of assistance in multiplication. His crowning achievement, however, was the invention of logarithms, to which he devoted fully twenty years of his life.

John Napier

122. Steps preceding computation. Before computation by means of the table can be taken up, two processes requiring considerable explanation and practice must be mastered.

I. *To find from the table the logarithm of a given number.*

II. *To find from the table the number corresponding to a given logarithm.*

123. Characteristic and mantissa. Unless a number is an exact power of 10; its logarithm consists of an integer and a decimal.

This fact is illustrated in Exercises 37–50, page 239.

The integral part of a logarithm is called its characteristic.

The decimal part of a logarithm is called its mantissa.

Log 800 = 2.903. Here 2 is the characteristic and .903 is the mantissa.

The characteristic of any number is obtained not from a table of logarithms but by an inspection of the number itself, according to rules which will now be derived.

$10^4 = 10{,}000$; that is, the log 10,000 = 4.
$10^3 = 1000$; that is, the log 1000 = 3.
$10^2 = 100$; that is, the log 100 = 2.
$10^1 = 10$; that is, the log 10 = 1.
$10^0 = 1$; that is, the log 1 = 0.
$10^{-1} = .1$; that is, log .1 = − 1.
$10^{-2} = .01$; that is, the log .01 = − 2.
$10^{-3} = .001$; that is, the log .001 = − 3.

The preceding statement indicates between what two integers the logarithm of a number less than 10,000 lies. This determines the characteristic.

Since 347 lies between 100 and 1000 (that is, between 10^2 and 10^3), log 347 must lie between 2 and 3 and must equal 2 (characteristic) plus a decimal (mantissa). Mantissas are always obtained from a table like that on page 244.

And since .0037 lies between .001 and .01 (that is, 10^{-3} and 10^{-2}), log .0037 $= -3$ plus a positive decimal or -2 plus a negative decimal.

For the determination of the characteristic of a positive number we have the following rules:

I. *The characteristic of a number greater than 1 is one less than the number of digits to the left of the decimal point.*

II. *The characteristic of a number less than 1 is negative and numerically one greater than the number of zeros between the decimal point and the first significant figure.*

Accordingly the characteristic of 2874 is 3; of 8 is 0; of .3 is -1; of .046 is -2; of .0078 is -3.

ORAL EXERCISES

What is the characteristic of:

1. 418.	5. 7300.	9. .73.
2. 7863.	6. 730.	10. .073.
3. 79.	7. 73.	11. .0073.
4. 8.	8. 7.3.	12. .00073.

13. 71.067. 15. .60000.

14. .000091. 16. .348.

The table on pages 244–245 gives the mantissa of numbers from 100 to 999. Before each mantissa a decimal point is understood.

The numbers 7860, 786, 78.6, 7.86, .0786, and .00786 are spoken of as composed of the same significant digits in the same order. They differ only in the position of the decimal point, and consequently their logarithms to the base 10 will have different characteristics, but they will have the same mantissa.

The last two points are easily illustrated by any two numbers which have the same significant digits in the same order.

$$8.26 = 10^{.917}, \text{ that is, } \log 8.26 = .917$$
$$8.26 \cdot 10^2 = 826 = 10^{.917} \cdot 10^2 = 10^{2.917}$$
$$\text{or } \log 826 = 2.917.$$

The property just explained does not belong to a system of logarithms in which the base is any number other than 10. Thus, if the base is 100, the most convenient number after 10, the logarithms of 7860, 786, 78.6, and 7.86 are respectively 1.9477, 1.4477, .9477, and .4477. While a certain regularity in characteristic and mantissa can be seen here, it is obvious that the rules for obtaining them would not be so simple as they are for the base 10. Moreover, it can be seen from the illustration just given that tables of a given accuracy are far shorter with the base 10 than they would be with any other base.

124. Use of the table. The table on page 244 gives the mantissas of all numbers containing one, two, or three significant digits. To obtain the logarithms of a number containing three or fewer significant digits we have the

RULE. *Determine the characteristic by inspection. Find in column "N" the first two significant figures of the given number. In the row with these and in the column headed by the third figure of the number, find the required mantissa.*

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396	8

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

ORAL EXERCISES

Find the logarithm of:

1. 74.	5. 63.	9. 500.
2. 381.	6. 21.	10. 97.
3. 485.	7. 5.	11. 738.
4. 739.	8. 50.	12. .487.

Solution. The characteristic of .487 is -1 and the mantissa is .6875. Hence $\log .487 = -1 + .6875$. This is often written in the abbreviated form, $\bar{1}.6875$. The mantissa is always kept positive in order to avoid the addition and subtraction of both positive and negative decimals, which in ordinary practice contain from three to five figures. Negative characteristics, being integers, are comparatively easy to take care of. (The student should note that log .487 is really negative, being $-1 + .6875$, or $-.3125$.)

13. .638.	17. 63800.
14. .0782.	18. 27000.
15. .00964.	19. 3.75.
16. .0307.	20. .784.

125. Interpolation. If a number contains four or more significant digits, its mantissa is not given in the table on page 244. It is possible, however, to calculate the mantissas of four- and five-digit numbers from mantissas which are given in the table. This process or the reverse of it is called interpolation.

If we desire the logarithm of a number not in the table, say 3575, we know that it lies between the logarithms of 3570 and 3580, which are given in the table. Since 3575 is halfway between 3570 and 3580, we assume, though it is only approximately true, that the required logarithm is

halfway between their logarithms, 3.5527 and 3.5539. In order to find log 3575 we first look up log 3570 and log 3580 and then take half (or .5) their difference (this difference may usually be taken from the column headed D) and add it to log 3570. This gives

$$\log 3575 = 3.5527 + .5 \times .0012 = 3.5533$$

Were we finding log 3578 we should take .8 of the difference between log 3570 and log 3580 and add it to log 3570 as follows:

$$\begin{aligned} \log 3578 &= 3.5527 + .8 \times .0012 \\ &= 3.5527 + .00096 \\ &= 3.5527 + .0010 \\ &= 3.5537 \end{aligned}$$

Observe that in using four-place tables one should not carry results to five figures. If the fifth figure of the logarithm is 5, 6, 7, 8, or 9, omit it and increase the fourth figure by 1; that is, *obtain results to the nearest figure in the fourth place.*

For finding the logarithm of a number not given in the table we have the

RULE. ***Prefix the proper characteristic to the mantissa of the first three significant figures.***

Then multiply the difference between this mantissa and the next greater mantissa in the table (called the tabular difference) by the remaining figures of the number preceded by a decimal point.

Add the product to the logarithm of the first three figures, taking the nearest decimal in the fourth place.

In this method of interpolation we have assumed that the increase in the logarithm is directly proportional to the

increase in the number. As has been said, this is not strictly true, yet the results thus obtained are nearly always correct to the fourth decimal place.

EXERCISES

Find the logarithm of:

1. 4785.	5. 3.528.	9. 46.209.
2. 6334.	6. 981.6.	10. .01877.
3. 42.67.	7. 42.83.	11. .0004444.
4. 96.42.	8. 8731.	12. .034567.

13. 3.1416.
14. 2.1782.
15. 3468000.
16. 19635.

126. Antilogarithms. An antilogarithm is the number corresponding to a given logarithm. Thus antilog 3 equals 1000.

If we desire the antilogarithm of a given logarithm, say 4.5132, we proceed as follows: The mantissa .5132 is found in the row which has 32 in column *N* and in the column which has 6 at the top. Hence the first three significant figures of the antilogarithm are 326. Since the characteristic is 4, the number must have five digits to the left of the decimal point. Thus antilog 4.5132 = 32600.

Therefore, if the mantissa of a given logarithm is found in the table, its antilogarithm is obtained by the

RULE. *Find the row and the column in which the given mantissa lies. In the row found take the two figures which are in column "N" for the first two significant figures of the antilogarithm and for the third figure the number at the top of the column in which the mantissa stands.*

Place the decimal point as indicated by the characteristic.

ORAL EXERCISES

Find the antilogarithm of:

1. 2.9304. 3. .5635. 5. $\bar{2}$.4502.
2. 1.7839. 4. $\bar{1}$.6284. 6. 5.8904.

7. 8.6222 − 10.

HINT. 8.6222 − 10 = − 2 + .6222.

8. 7.8261 − 10. 10. 4.1335 − 7.
9. 6.5877. 11. − 3 + .6304.

12. .9053. 13. 7.9961 − 10. 14. 9.7443 − 10.

If the mantissa of a given logarithm, as in 1.4571, is not given in the table, the antilogarithm is obtained by interpolation as follows:

The mantissa .4571 lies between

.4564, the mantissa of the sequence 286,

and .4579, the mantissa of the sequence 287.

Therefore the antilogarithm of 1.4571 lies between 28.6 and 28.7. Since the tabular difference is 15 and the difference between .4564 and .4571 is 7, the mantissa .4571 lies $\frac{7}{15}$ of the way from .4564 to .4579. Therefore the required antilogarithm lies $\frac{7}{15}$ of the way from 28.6 to 28.7.

Then antilog 1.4571 = 28.6 + $\frac{7}{15}$(.1)

and 28.6 + .046 = 28.65.

Therefore to find the number corresponding to a given mantissa when the mantissa is not found in the table we have the

RULE. *Write the number of three figures corresponding to the lesser of two mantissas between which the given mantissa lies.*

Subtract the less mantissa from the given one and divide the remainder by the tabular difference to two decimal places. If the second digit is 5 *or more, increase the first digit by* 1; *if less than* 5, *omit it.*

Annex the resulting digit to the three already found and place the decimal point where indicated by the characteristic.

EXERCISES

Find the antilogarithms of:

1. 1.4860.	5. $\bar{1}.3626$.	9. .6187.
2. 2.4796.	6. $9.8448 - 10$.	10. $7.5257 - 10$.
3. 4.9481.	7. $6.0748 - 9$.	11. 8.4230.
4. 0.3727.	8. $5.8153 - 10$.	12. $8.6510 - 10$.

127. Multiplication. Multiplication by logarithms depends on the

THEOREM. *The logarithm of the product of two numbers is the sum of the logarithms of the numbers.*

That is, for the numbers x and z

$$\log_b(x \cdot z) = \log_b x + \log_b z.$$

Proof. Let $\log_b x = l_1$ (1)

and $\log_b z = l_2$. (2)

From (1), $x = b^{l_1}$. (3)

From (2), $z = b^{l_2}$. (4)

(3) $\times$ (4), $xz = b^{l_1+l_2}$. (5)

Therefore, $\log_b xz = l_1 + l_2$

$= \log_b x + \log_b z.$

It follows from this theorem that the logarithm of the product of more than two numbers is the sum of the logarithms of the factors. Thus $\log (x \cdot y \cdot z) = \log x + \log y + \log z$.

EXERCISES

Perform the indicated operation by logarithms:

1. 16×35.

Solution.

$$\log 16 = 1.2041$$
$$\log 35 = 1.5441$$

Adding, $\log (16 \times 35) = 2.7482$

Antilog $2.7482 = 560$

2. 27×38.
3. 65×79.
4. 318×7.
5. 7.6×4.91.
6. 231×47.
7. 873×67.
8. 769×257.
9. 7860×1325.
10. 64.37×3.142.
11. 2600×318.
12. 3718×5694.
13. $3470 \times .0485$.

Solution. $\log (3470 \times .0485) = 3.5403 + 8.6857 - 10$
$= 12.2260 - 10 = 2.2260$

Antilog $2.2260 = 168.3$.

Since the mantissa is always positive, any number carried over from the tenths' column to the units' column is positive. This occurs in the preceding solution, where $.6 + .5 = 1.^{+}$, giving $+ 1$ to be added to the sum of the characteristics $+ 3$ and $- 2$, in the units' column. Mistakes in such cases will be less frequent if the logarithms with negative characteristics be written as in the 8–10 notation.

14. $438 \times .725$.
15. $637 \times .0354$.
16. $91.7 \times .00637$.
17. $.00842 \times .6071$.

18. $(4316)(-3.81)$

HINT. Determine by inspection the sign of the product. Then calculate as if all signs were positive and give to the result the proper sign.

19. $23(-284)$.

20. $4.2 \times 96 \times 7.1$.

21. $4.83 \times 2.36 \times .971$.

22. $57(-28)(-3.65)$.

128. Division. Division by logarithms depends on the

THEOREM. ***The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.***

That is, for the numbers x and z

$$\log_b \frac{x}{z} = \log_b x - \log_b z.$$

Proof. Let $\log_b x = l_1$ (1)

and $\log_b z = l_2$. (2)

From (1), $x = b^{l_1}$. (3)

From (2), $z = b^{l_2}$. (4)

(3) ÷ (4), $\frac{x}{z} = b^{l_1 - l_2}$.

Therefore, $\log_b \frac{x}{z} = l_1 - l_2$

$= \log_b x - \log_b z.$

It follows from the multiplication and the division theorems that if the product of two or more numbers is divided by the product of two or more others, the logarithm of the resulting quotient is the sum of the logarithms of the factors of the dividend minus the sum of the logarithms of the factors of the divisor. Thus

$$\log \frac{abc}{xyz} = \log a + \log b + \log c - (\log x + \log y + \log z).$$

EXERCISES

Using logarithms, perform the indicated operations:

1. $765 \div 34$.

Solution.
$$\log 765 = 2.8837$$
$$\log 34 = 1.5315$$
Subtracting, $$\log (765 \div 34) = 1.3522$$
$$\text{antilog } 1.3522 = 22.5$$

2. $625 \div 25$. 3. $784 \div 37$. 4. $563 \div 4.27$.

5. $4380 \div 74.3$. 7. $468.4 \div 47.2$.

6. $79.43 \div 3.25$. 8. $3754 \div 2869$.

9. $4.38 \div .0735$.

Solution.
$$\log 4.38 = .6415 = 10.6415 - 10$$
$$\log .0735 = 8.8663 - 10$$
$$\log (4.38 \div .0735) = 1.7752$$
$$\text{Antilog } 1.7752 = 59.6$$

10. $3.76 \div .0759$.
11. $4.23 \div .687$.
12. $25.37 \div .00786$.
13. $.738 \div .0381$.
14. $.0156 \div .492$.
15. $.0864 \div 73.52$.
16. $.0693 \div 42.86$.
17. $43 \times 728 \div 35.24$.
18. $87.2 \times 43.9 \div 28.46$.
19. $\dfrac{468 \times 75.34}{381 \times 27}$.
20. $\dfrac{497(.7852)}{3.142}$.
21. $\dfrac{439(-8.64)3056}{(-75.3)(-8.61)}$.

129. Raising to a power by means of logarithms. Raising to a power by means of logarithms depends on the

THEOREM. *The logarithm of the rth power of a number is r times the logarithm of the number.*

That is, for the numbers r and x, $\log_b x^r = r \log_b x$.

Proof. Let $\log_b x = l.$ (1)

Then $x = b^l.$ (2)

Raising both members of (2) to the rth power,

$$x^r = b^{rl}. \quad (3)$$

Therefore $\log_b x^r = rl.$ (4)

From (1) and (4) $\log_b x^r, \quad = r \log_b x.$

EXERCISES

Compute, using logarithms:

1. $(3.42)^4$.

Solution. $\log 3.42 = .5340.$

$\log (3.42)^4 = 4(.5340) = 2.1360.$

Therefore $(3.42)^4 = \text{antilog } 2.1360 = 136.8.$

2. $(4.59)^3$. **4.** $(3.812)^4$.

3. $(47.91)^2$. **5.** $(.0738)^3$.

Solution. $\log .0738 = \bar{2}.8681 = 8.8681 - 10$

$\log (.0738)^3 = 3(\bar{2}.8681) = 3(8.8681 - 10)$

$= \bar{4}.6043 = 26.6043 - 30.$

Therefore $(.0738)^3 = \text{antilog } \bar{4}.6043 = .0004021.$

6. $(.0874)^4$. **7.** $(.007495)^2$. **8.** $(.9284)^3$.

9. $6984(842)^2$.

10. $482(31.54)^2$.

11. $\dfrac{(43)^2(8381)^3}{(.097)^2(435)^4}$.

12. $.967 \div (.0723)^3$.

130. Extracting a root by means of logarithms. Extracting a root by means of logarithms depends on the

THEOREM. *The logarithm of the real rth root of a number is the logarithm of the number divided by r.*

That is, for the real numbers r and n, $\log_b \sqrt[r]{n} = \frac{1}{r} \log_b n$.

Proof. Let $\log_b n = l.$ (1)

Then $n = b^l.$ (2)

Extracting the rth root of both members of (2),

$$(n)^{\frac{1}{r}} = (b^l)^{\frac{1}{r}} = b^{\frac{l}{r}}. \quad (3)$$

Therefore $\log_b (n)^{\frac{1}{r}} = \frac{l}{r}.$

EXERCISES

Compute, using logarithms:

1. $\sqrt[3]{495}$.

Solution. $\log 495 = 2.6946.$

$\log \sqrt[3]{495} = \frac{1}{3}(2.6946) = .8982.$

Therefore $\sqrt[3]{495} = \text{antilog } .8982 = 7.91.$

2. $\sqrt[3]{876}$.

3. $\sqrt[3]{1925}$.

4. $\sqrt[4]{1732}$.

5. $\sqrt[3]{.0785}$.

Solution. $\log .0785 = \bar{2}.8949.$

If one divided $\bar{2}.8949$ as it stands by 3, he would be likely to confuse the negative characteristic and the positive mantissa. This and other difficulties may be easily avoided by adding to the characteristic and subtracting from the resulting logarithm any integral multiple of the index of the root which will make the characteristic positive.

Thus $\log .0785 = 7.8949 - 9.$

Dividing by 3,

$\log \sqrt[3]{.0785} = 2.6316 - 3.$

Therefore $\sqrt[3]{.0785} = \text{antilog } \bar{1}.6316 = .4282.$

6. $\sqrt{.0853}$.
7. $\sqrt[3]{.0007625}$.
8. $\sqrt[4]{.005937}$.
9. $\sqrt[5]{.7648}$.
10. $(5.37)^{\frac{2}{3}}$.
11. $(5.368)^{\frac{3}{2}}$.
12. $(-7.953)^{\frac{4}{3}}$.
13. $\sqrt{49.5(7.38)^2}$.
14. $\sqrt[3]{(.0395)^2}$.
15. $\sqrt[7]{361}$.
16. $\sqrt{97}\sqrt[3]{742}$.
17. $\frac{32}{518} \cdot \frac{\sqrt{431}}{217}\sqrt[3]{792}$.

18. Determine the logarithms of 4267, 426.7, 42.67, and 4.267 to the base 10 and to the base 100. Compare the results. What fact about logarithms do these results emphasize?

NOTE. The preceding four-place table will usually give results correct to one half of one per cent. Five-place tables give the mantissa to five decimal places of the numbers from 1 to 9999 and, by interpolation, the mantissa of numbers from 1 to 99,999. Such tables give results correct to one twentieth of one per cent, a degree of accuracy which is sufficient for most engineering work.

Six-place tables give the mantissa to six decimals for the same range of numbers as a five-place table, but the labor of using a six-place table is much greater than that of using a five-place one.

Seven-place tables contain the mantissas of the numbers from 1 to 99,999. Such tables are needed in certain kinds of engineering work and are of constant use in astronomy.

In place of a table of logarithms engineers often use an instrument called a slide rule. This is really a mechanical table of logarithms arranged ingeniously for rapid practical use. Results can be obtained with such an instrument far more quickly than with an ordinary table of logarithms, and that without recording or even thinking of a single logarithm. A slide rule ten inches long usually gives results correct to three figures. For work requiring greater precision larger and more elaborate instruments which give a ten-figure accuracy are used.

131. Exponential equations. An exponential equation is an equation in which the unknown occurs in an exponent.

Many exponential equations are readily solved by means of logarithms, since $\log a^x = x \log a$.

Thus let $a^x = c$. Then $x \log a = \log c$. Whence $x = \log c \div \log a$.

EXAMPLE

Solve for x: $7^x = 283$.

Solution. $\log 7^x = \log 283$

or $x \log 7 = \log 283$.

Whence $x = \dfrac{\log 283}{\log 7} = \dfrac{2.4518}{.8451} = 2.901.$

EXERCISES

Solve for x:

1. $4^x = 31$. 2. $7^x = 5$. 3. $18^{2x} = 43$.

4. $9^{x+1} = 43$. 6. $5^{2x-1} = 81$.

5. $3 = 1.05^x$. 7. $(.75)^x = 52$.

Solution. $x \log .75 = \log 52$.

$$x = \frac{\log 52}{\log .75} = \frac{1.7160}{1.8751} = \frac{1.7160}{-1 + .8751} = \frac{1.7160}{-.1249} = -13.74.$$

8. $(.0863)^x = .284$. 10. $(.0073)^{x+1} = (.084)^2$.

9. $(242)^{1-x} = 500$. 11. $(72)^{2x-3} = (28)^x$.

MISCELLANEOUS EXERCISES

1. Prove $\log ab = \log a + \log b$.

2. Prove $\log \dfrac{a}{b} = \log a - \log b$.

3. Prove $\log (a)^n = n \log a$.

4. Prove $\log \sqrt[n]{a} = \dfrac{1}{n} \log a$.

5. Does the logarithm of a negative number to a positive base exist? Explain.

Find, without using the table, the numeric values of:

6. $\log_2 4$.
7. $\log_5 25$.
8. $2 \log_3 27$.
9. $\log_9 27$.
10. $4 \log_5 125$.
11. $\log_4 8 + \log_8 4$.
12. $\log_8 16 + 4 \log_{16} 8$.
13. $4 \log_{27} 81 + 5 \log_{81} 27$.
14. $4 \log_5 125 - 3 \log_{25} 5 + 2 \log_5 25$.
15. $5 \log_2 (\frac{1}{2}) + 3 \log_4 (\frac{1}{64}) + 7 \log_8 4$.

Simplify:

16. $\log \frac{4}{9} + \log \frac{27}{48}$.
17. $\log \frac{17}{27} - \log \frac{34}{63}$.
18. $\log \frac{35}{9} + \log \frac{15}{14} - \log \frac{6}{25}$.
19. $3 \log 5 + 5 \log 3$.
20. $2 \log 9 + 3 \log 5 - 7 \log 15$.

Show that

21. $\log \sqrt{a^2 - x^2} = \frac{1}{2} \log (a + x) + \frac{1}{2} \log (a - x)$.

22. $\log \dfrac{a^3 + x^3}{a} = \log (a + x) + \log (a^2 - ax + x^2) - \log a$.

23. $\log \sqrt{\dfrac{s(s - b)(s - c)}{s - a}} =$

$\frac{1}{2}[\log s + \log (s - b) + \log (s - c) - \log (s - a)]$.

In the following obtain results to four figures:

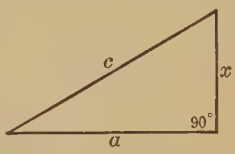
24. The circumference of a circle is $2 \pi R$. ($\pi = 3.1416$.) Find the circumference of a circle whose radius is 23.15.

25. The area of a circle is πR^2. Find the area of a circle whose diameter is 18.76 inches.

26. The volume of a sphere is $\frac{4}{3}\pi R^3$. Find the volume of a sphere whose diameter is .7854.

27. The area of the surface of a sphere is $4\pi R^2$. Find the area of the earth's surface, the radius being 3959 miles.

If two sides of a right triangle are given and one of the two is the hypotenuse, the other side can be computed by logarithms. Let c be the hypotenuse and a the other given side, then if x is the unknown side, $x^2 = c^2 - a^2$



or $x = \sqrt{c^2 - a^2} = \sqrt{(c + a)(c - a)}$

or $\log x = \frac{1}{2}\log (c + a) + \frac{1}{2}\log (c - a)$.

28. The hypotenuse of a right triangle is 435 and one other side is 347. Find the third side.

29. Find the number of digits in: (*a*) 2^{43}; (*b*) $(3)^{200}$.

30. In how many years will one dollar double itself at four per cent compound interest?

Solution. At the end of one year the amount of one dollar at four per cent is one dollar and four cents; at the end of two years it is $(1.04)(1.04)$, or $(1.04)^2$; at the end of three years it is $(1.04)^3$, and at the end of x years it is $(1.04)^x$.

If x is the number of years required, $(1.04)^x = 2$.

Taking the logarithms of both members of the equation,

$$x \log 1.04 = \log 2$$

Solving, $$x = \frac{\log 2}{\log 1.04} = \frac{.3010}{.0170} = 17.70.$$

In making computations of this nature by the aid of logarithms, care must be exercised not to retain more significant figures in the result than are given with accuracy by the process.

31. In how many years will one dollar double itself at five per cent interest compounded annually?

32. In how many years will any sum of money treble itself at four and one fourth per cent interest compounded annually?

33. About 300 years ago the Dutch paid twenty-four dollars for the island of Manhattan. At four per cent interest compounded annually, what would this payment amount to at the present time?

34. In how many years will one dollar double itself at four per cent interest compounded semi-annually?

35. What will two thousand five hundred dollars amount to in 12 years at 4 per cent interest compounded quarterly?

36. Show that the amount of P dollars in t years at r per cent interest compounded annually is $P(1+r)^t$; compounded semi-annually is $P\left(1+\frac{r}{2}\right)^{2t}$; compounded quarterly is $P\left(1+\frac{r}{4}\right)^{4t}$; and compounded monthly is $P\left(1+\frac{r}{12}\right)^{12t}$.

Solve for x:

37. $a^x = m^{x-1}$.

38. $a^{x+1} = m^{x-2}$.

39. $a^{x+1} \cdot m^x = c^{3x}$.

40. $A = P(1+r)^x$.

41. $m^{x^2+2x} = n$.

42. $3^x \cdot 2^{\frac{1}{x}} = 6$.

43. $a^{4x} + 8\,a^{2x} = 6\,a^{3x}$.

44. $a^{5x} + a^{4x} = 6\,a^{4x} - 6\,a^{3x}$.

45. Show that $a^{\log_a x} = x$.

46. Show that $e^{\log_e (x^2-3x)} = x^2 - 3x$.

NOTE. It is not a little remarkable that just at the time when Galileo and Kepler were turning their attention to the laborious computation of the orbits of planets, Napier should be devising a method which simplifies these processes. It was said a hundred years ago, before astronomical computations became so complex as they now are, that the invention of logarithms, by shortening the labors, doubled the effective life of the astronomer. To-day the remark is well inside the truth.

In the presentation of the subject in modern textbooks a logarithm is defined as an exponent. But it was not from this point of view that they were first considered by Napier. In fact it was not till long after his time that the theory of exponents was understood clearly enough to admit of such application. This relation was noticed by the mathematician Euler, about one hundred and fifty years after logarithms were invented.

It was by a comparison of the terms of certain arithmetical and geometric progressions that Napier derived his logarithms. They were not exactly like those used commonly to-day, for the base which Napier used was not 10. Soon after the publication (1614) of Napier's work, Henry Briggs, an English professor, was so much impressed with its importance that he journeyed to Scotland to confer with Napier about the discovery. It is probable that they both saw the necessity of constructing a table for the base 10, and to this enormous task Briggs applied himself. With the exception of one gap, which was filled in by another computer, Briggs's tables form the basis for all the common logarithms which have appeared from that day to this.

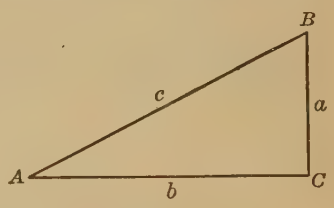
CHAPTER XVIII

TRIGONOMETRY

132. Introduction. In surveying, astronomy, and in countless other applications of science it is necessary to solve triangles. A triangle has six parts, three sides and three angles. When three of these parts, of which one at least must be a side, have been given or have been measured, the other three can be computed by the methods of trigonometry. In developing the principles of trigonometry it is simplest to begin with the right triangle.

In order to attack the problem stated above it is necessary to denote the ratios of the sides of a right triangle by certain names.

Let ABC be any right triangle with the right angle at C.



The sine of the angle $A = \frac{a}{c} = \frac{\text{side opposite}}{\text{hypotenuse}}$.

The cosine of the angle $A = \frac{b}{c} = \frac{\text{side adjacent}}{\text{hypotenuse}}$.

The tangent of the angle $A = \frac{a}{b} = \frac{\text{side opposite}}{\text{side adjacent}}$.

The cotangent of angle $A = \frac{b}{a} = \frac{\text{side adjacent}}{\text{side opposite}}$.

These definitions are abbreviated as follows:

$$\sin A = \frac{a}{c},\ \cos A = \frac{b}{c},\ \tan A = \frac{a}{b},\ \cot A = \frac{b}{a}.$$

The values of the trigonometric ratios for all angles have been computed to many decimal places. The table on page 265 gives the values of the sine, the cosine, and the tangent correct to four decimal places for every degree from zero to ninety degrees.

EXERCISES

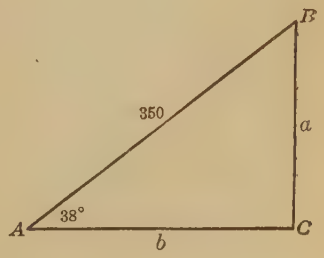
1. Read from the table:

sin 31°	sin 68°	cos 18°	cos 72°
tan 27°	tan 82°	cot 40°	cot 73°

2. From the table determine the number of degrees in angle x if:

(a) $\sin x = .9962$.
(b) $\sin x = .5150$.
(c) $\cos x = .9135$.
(d) $\cos x = .4848$.
(e) $\tan x = 3.7321$.
(f) $\tan x = .5774$.
(g) $\cot x = 1.8807$.
(h) $\cot x = .6009$.
(i) $\cos x = .8572$.
(j) $\sin x = .9994$.
(k) $\cot x = 57.290$.

3. In the right triangle ABC, side $c = 350$ and angle $A = 38°$. Find the other parts.



HINT. $\sin A = \frac{a}{c}$.

$$\sin 38° = \frac{a}{350}.$$

$$a = 350(\sin 38°)$$
$$= 350(.6157) = 215.5.$$

Find the missing parts in the following right triangles. Perform the multiplications and divisions as in Exercise 3 above, or by logarithms if the teacher prefers.

	a	b	c	A	B
4.	?	?	300	18°	?
5.	120	?	?	?	56°
6.	?	225	?	67°	?
7.	?	86.6	100	?	?
8.	707.1	?	1000	?	?
9.	40	20.38	?	?	?
10.	376.8	500	?	?	?
11.	17.05	?	25	?	?
12.	?	73.08	80	?	?
13.	606	1500	?	?	?
14.	125	50.5	?	?	?
15.	?	67.84	80	?	?
16.	1386	?	?	78°	?
17.	?	214	?	?	53°
18.	10.45	99.45	?	?	?

133. Interpolation. The word "interpolation" means literally a placing between. The process consists in finding from the numbers in a given table a number not listed there. It applies to any table, one of square roots, cube roots, or logarithms, or a trigonometric table, such as that given on page 265.

There are two problems of interpolation in a trigonometric table:

(*a*) *Given an* **angle** *not in the table, to find the corresponding sine, cosine, tangent, or cotangent.*

(*b*) *Given the* **value** *of a sine, cosine, tangent, or cotangent not in the table, to find the corresponding angle.*

Four-place Trigonometric Table

Angle	sin	cos	tan	cot	Angle	sin	cos	tan	cot
0°	.0000	1.0000	.0000	∞	45°	.7071	.7071	1.0000	1.0000
1°	.0175	.9998	.0175	57.290	46°	.7193	.6947	1.0355	.9657
2°	.0349	.9994	.0349	28.636	47°	.7314	.6820	1.0724	.9325
3°	.0523	.9986	.0524	19.081	48°	.7431	.6691	1.1106	.9004
4°	.0698	.9976	.0699	14.300	49°	.7547	.6561	1.1504	.8693
5°	.0872	.9962	.0875	11.430	50°	.7660	.6428	1.1918	.8391
6°	.1045	.9945	.1051	9.5144	51°	.7771	.6293	1.2349	.8098
7°	.1219	.9925	.1228	8.1443	52°	.7880	.6157	1.2799	.7813
8°	.1392	.9903	.1405	7.1154	53°	.7986	.6018	1.3270	.7536
9°	.1564	.9877	.1584	6.3138	54°	.8090	.5878	1.3764	.7265
10°	.1736	.9848	.1763	5.6713	55°	.8192	.5736	1.4281	.7002
11°	.1908	.9816	.1944	5.1446	56°	.8290	.5592	1.4826	.6745
12°	.2079	.9781	.2126	4.7046	57°	.8387	.5446	1.5399	.6494
13°	.2250	.9744	.2309	4.3315	58°	.8480	.5299	1.6003	.6249
14°	.2419	.9703	.2493	4.0108	59°	.8572	.5150	1.6643	.6009
15°	.2588	.9659	.2679	3.7321	60°	.8660	.5000	1.7321	.5774
16°	.2756	.9613	.2867	3.4874	61°	.8746	.4848	1.8040	.5543
17°	.2924	.9563	.3057	3.2709	62°	.8829	.4695	1.8807	.5317
18°	.3090	.9511	.3249	3.0777	63°	.8910	.4540	1.9626	.5095
19°	.3256	.9455	.3443	2.9042	64°	.8988	.4384	2.0503	.4877
20°	.3420	.9397	.3640	2.7475	65°	.9063	.4226	2.1445	.4663
21°	.3584	.9336	.3839	2.6051	66°	.9135	.4067	2.2460	.4452
22°	.3746	.9272	.4040	2.4751	67°	.9205	.3907	2.3559	.4245
23°	.3907	.9205	.4245	2.3559	68°	.9272	.3746	2.4751	.4040
24°	.4067	.9135	.4452	2.2460	69°	.9336	.3584	2.6051	.3839
25°	.4226	.9063	.4663	2.1445	70°	.9397	.3420	2.7475	.3640
26°	.4384	.8988	.4877	2.0503	71°	.9455	.3256	2.9042	.3443
27°	.4540	.8910	.5095	1.9626	72°	.9511	.3090	3.0777	.3249
28°	.4695	.8829	.5317	1.8807	73°	.9563	.2924	3.2709	.3057
29°	.4848	.8746	.5543	1.8040	74°	.9613	.2756	3.4874	.2867
30°	.5000	.8660	.5774	1.7321	75°	.9659	.2588	3.7321	.2679
31°	.5150	.8572	.6009	1.6643	76°	.9703	.2419	4.0108	.2493
32°	.5299	.8480	.6249	1.6003	77°	.9744	.2250	4.3315	.2309
33°	.5446	.8387	.6494	1.5399	78°	.9781	.2079	4.7046	.2126
34°	.5592	.8290	.6745	1.4826	79°	.9816	.1908	5.1446	.1944
35°	.5736	.8192	.7002	1.4281	80°	.9848	.1736	5.6713	.1763
36°	.5878	.8090	.7265	1.3764	81°	.9877	.1564	6.3138	.1584
37°	.6018	.7986	.7536	1.3270	82°	.9903	.1392	7.1154	.1405
38°	.6157	.7880	.7813	1.2799	83°	.9925	.1219	8.1443	.1228
39°	.6293	.7771	.8098	1.2349	84°	.9945	.1045	9.5144	.1051
40°	.6428	.7660	.8391	1.1918	85°	.9962	.0872	11.430	.0875
41°	.6561	.7547	.8693	1.1504	86°	.9976	.0698	14.300	.0699
42°	.6691	.7431	.9004	1.1106	87°	.9986	.0523	19.081	.0524
43°	.6820	.7314	.9325	1.0724	88°	.9994	.0349	28.636	.0349
44°	.6947	.7193	.9657	1.0355	89°	.9998	.0175	57.290	.0175
45°	.7071	.7071	1.0000	1.0000	90°	1.0000	.0000	∞	.0000

The process of interpolation depends on the assumption that a small change in the ratio is proportional to the change in the corresponding angle for the degree intervals.

EXERCISES

1. Find sin 26° 30′ and sin 26° 20′.

Solution. From the table sin 26° = .4384 and sin 27° = .4540. Sin 26° 30′ lies halfway between these two numbers or at .4462. Sin 26° 20′ would be one third of the way from .4384 to .4540 or at .4436.

2. Find A if sin A = .3365.

Solution. Reference to the table shows that .3365 lies between .3256 and .3420. Hence A must lie between 19° and 20°. The entire difference between .3420 and .3256 is .0164, which corresponds to a change of one degree or 60 minutes. But .3365 is .0109 greater than .3256. Hence .0109 corresponds to $\frac{109}{164} \times 60$ or 40 minutes of the difference between 19° and 20°. Therefore angle A = 19° 40′.

3. Find the value of the function for:

(*a*) sin 27° 20′. (*d*) tan 18° 40′. (*g*) sin 71° 24′.
(*b*) sin 43° 50′. (*e*) tan 82° 20′. (*h*) tan 18° 18′.
(*c*) sin 69° 10′. (*f*) tan 64° 48′. (*i*) sin 24° 24′.

4. Find the angle A if:

(*a*) sin A = .4924. (*g*) tan A = .7046.
(*b*) sin A = .9304. (*h*) tan A = .4417.
(*c*) sin A = .3638. (*i*) sin A = .2108.
(*d*) sin A = .7030. (*j*) sin A = .9652.
(*e*) tan A = .8355. (*k*) tan A = .0787.
(*f*) tan A = .8591. (*l*) tan A = 2.1775.

NOTE. The sine and the tangent increase as the angle increases, while the cosine and the cotangent decrease as the angle increases. A moment's inspection of the definitions and the triangle of § 132 will verify this extremely important fact, and a glance at the table on page 265 will further confirm it. Hence it is important to observe that while the process of interpolation is similar in the two cases it is not identical.

5. Find cos 18° 40′.

Solution. From the table cos 18° = .9511 and cos 19° = .9455. But 40′ corresponds to $\frac{40}{60}$ of .0056 = .0037. Therefore cos 18° 40′ = .9511 minus .0037 = .9474.

6. Find A if cot A = 1.1237.

Solution. Reference to the table shows that 1.1237 lies between 1.1504 the cot 41° and 1.1106 the cot 42°. The difference between these two cotangents is .0398. The difference between 1.1504 and 1.1237 is .0267 and $\frac{267}{398}$ of 60′ = 40′. Therefore angle A = 41° 40′.

7. Find the value of the function for:

(*a*) cos 32° 40′. (*d*) cot 35° 10′. (*g*) cot 67° 20′.
(*b*) cos 23° 20′. (*e*) cos 41° 50′. (*h*) cot 38° 50′.
(*c*) cot 27° 40′. (*f*) cos 54° 18′. (*i*) cos 25° 10′.

8. Find the angle if:

(*a*) cos A = .8689.
(*b*) cos A = .9717.
(*c*) cos A = .2193.
(*d*) cos A = .8732.
(*e*) cot A = 1.7090.
(*f*) cot A = 2.3945.
(*g*) cot A = 1.
(*h*) cos A = 0.
(*i*) sin A = 0.
(*j*) tan A = 0.
(*k*) tan A = 1.
(*l*) cos A = .8465.
(*m*) cos A = .3886.
(*n*) cos A = .8355.
(*o*) cos A = .8021.
(*p*) cot A = 1.5900.
(*q*) cot A = .9217.
(*r*) cos A = .7790.

Find the missing parts in the following right triangles, using logarithms in the computation if the teacher prefers.

	a	b	c	A	B
9.	24	?	?	18° 40′	?
10.	?	625	?	?	40° 20′
11.	?	?	1200	51° 20′	?
12.	3	4	?	?	?
13.	12	5	?	?	?
14.	28	45	?	?	?
15.	?	55	73	?	?
16.	136	?	305	?	?
17.	1236	?	?	81° 40′	?
18.	4380	?	?	27° 50′	?

134. Logarithms of the trigonometric functions. Since the solution of triangles and other uses of trigonometry also require both multiplication and division, the actual labor involved is much shortened by using a table of the logarithms of the functions instead of the functions themselves. Such a table is given on page 270. Interpolation with such a table is precisely the same as with a table of natural functions.

In using the tables on page 270 it is important to note that the sines and cosines of all angles from zero to ninety degrees are less than one except cosine 0° and sin 90°. Also the tangents of all angles less than 45° and the cotangents of all angles greater than 45° are less than one. Hence the logarithms of these functions will be negative, and minus 10 is *understood* after each in the table.

On the other hand, the tangents of angles greater than 45° and the cotangents of angles less than 45° are greater than one, and hence their logarithms are *positive* and are printed in full.

The foregoing can be reduced to one simple statement:

Minus 10 is understood after all logarithms in the table except those in the column at the top of which is the word "COT."

EXERCISES

Solve the following, using the logarithms tables on page 270:

1. In the right triangle ABC side $AC = 9.86$ and $\angle A = 34° 20'$. Solve the triangle.

Solution. $\tan A = \frac{a}{b}$.

$$a = b \tan A.$$

Hence
$$\log a = \log b + \log \tan A$$
$$= \log 9.86 + \log \tan 34° 20'.$$

$$\log 9.86 = .9939$$
$$\log \tan 34°20' = 9.8344 - 10$$
$$\log a = .8283$$
$$a = 6.734.$$

$$\sin A = \frac{a}{c}$$

or
$$c = \frac{a}{\sin A}.$$

$$\log c = \log a - \log \sin A, \text{ etc.}$$

Compute the missing parts in the following right triangles.

	a	b	c	A	B
2.	187	?	?	41° 40′	?
3.	?	64.9	?	19° 20′	?
4.	?	?	.0785	?	35° 20′
5.	154	?	200	?	?
6.	609	481	?	?	?

Logarithms of Trigonometric Ratios

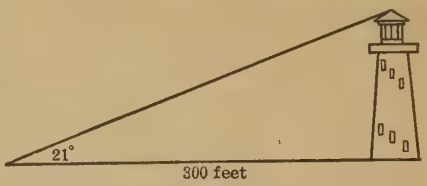
°	SIN	COS	TAN	COT		°	SIN	COS	TAN	COT	
0°	$-\infty$	10.0000	$-\infty$	∞	**90°**	**11°**	9.2806	9.9919	9.2887	0.7113	**79°**
15′	7.6398	.0000	7.6398	2.3602	45′	15′	.2902	.9916	.2987	.7013	45′
30′	.9408	.0000	.9409	.0591	30′	30′	.2997	.9912	.3085	.6915	30′
45′	8.1169	.0000	8.1170	1.8830	15′	45′	.3089	.9908	.3181	.6819	15′
1°	8.2419	9.9999	8.2419	1.7581	**89°**	**12°**	9.3179	9.9904	9.3275	0.6725	**78°**
15′	.3388	.9999	.3389	.6611	45′	15′	.3267	.9900	.3367	.6633	45′
30′	.4179	.9999	.4181	.5819	30′	30′	.3353	.9896	.3458	.6542	30′
45′	.4848	.9998	.4851	.5149	15′	45′	.3438	.9892	.3546	.6454	15′
2°	8.5428	9.9997	8.5431	1.4569	**88°**	**13°**	9.3521	9.9887	9.3634	0.6366	**77°**
15′	.5939	.9997	.5943	.4057	45′	15′	.3602	.9883	.3719	.6281	45′
30′	.6397	.9996	.6401	.3599	30′	30′	.3682	.9878	.3804	.6196	30′
45′	.6810	.9995	.6815	.3185	15′	45′	.3760	.9874	.3886	.6114	15′
3°	8.7188	9.9994	8.7194	1.2806	**87°**	**14°**	9.3837	9.9869	9.3968	0.6032	**76°**
15′	.7535	.9993	.7542	.2458	45′	15′	.3912	.9864	.4048	.5952	45′
30′	.7857	.9992	.7865	.2135	30′	30′	.3986	.9859	.4127	.5873	30′
45′	.8156	.9991	.8165	.1835	15′	45′	.4059	.9855	.4204	.5796	15′
4°	8.8436	9.9989	8.8446	1.1554	**86°**	**15°**	9.4130	9.9849	9.4281	0.5719	**75°**
15′	.8699	.9988	.8711	.1289	45′	15′	.4200	.9844	.4356	.5644	45′
30′	.8946	.9987	.8960	.1040	30′	30′	.4269	.9839	.4430	.5570	30′
45′	.9181	.9985	.9196	.0804	15′	45′	.4337	.9834	.4503	.5497	15′
5°	8.9403	9.9983	8.9420	1.0580	**85°**	**16°**	9.4403	9.9828	9.4575	0.5425	**74°**
15′	.9614	.9982	.9633	.0367	45′	15′	.4469	.9823	.4646	.5354	45′
30′	.9816	.9980	.9836	.0164	30′	30′	.4533	.9817	.4716	.5284	30′
45′	9.0008	.9978	9.0030	0.9970	15′	45′	.4597	.9812	.4785	.5215	15′
6°	9.0192	9.9976	9.0216	0.9784	**84°**	**17°**	9.4659	9.9806	9.4853	0.5147	**73°**
15′	.0369	.9974	.0395	.9605	45′	15′	.4721	.9800	.4921	.5079	45′
30′	.0539	.9972	.0567	.9433	30′	30′	.4781	.9794	.4987	.5013	30′
45′	.0702	.9970	.0732	.9268	15′	45′	.4841	.9788	.5053	.4947	15′
7°	9.0859	9.9968	9.0891	0.9109	**83°**	**18°**	9.4900	9.9782	9.5118	0.4882	**72°**
15′	.1011	.9961	.1045	.8955	45′	15′	.4958	.9776	.5182	.4818	45′
30′	.1157	.9963	.1194	.8806	30′	30′	.5015	.9770	.5245	.4755	30′
45′	.1299	.9960	.1338	.8662	15′	45′	.5071	.9763	.5308	.4692	15′
8°	9.1436	9.9958	9.1478	0.8522	**82°**	**19°**	9.5126	9.9757	9.5370	0.4630	**71°**
15′	.1568	.9955	.1614	.8387	45′	15′	.5181	.9750	.5431	.4569	45′
30′	.1697	.9952	.1745	.8255	30′	30′	.5235	.9743	.5491	.4509	30′
45′	.1822	.9949	.1873	.8127	15′	45′	.5288	.9737	.5551	.4449	15′
9°	9.1943	9.9946	9.1997	0.8003	**81°**	**20°**	9.5341	9.9730	9.5611	0.4389	**70°**
15′	.2061	.9943	.2118	.7882	45′	15′	.5392	.9723	.5669	.4331	45′
30′	.2176	.9940	.2236	.7764	30′	30′	.5443	.9716	.5727	.4273	30′
45′	.2288	.9937	.2351	.7649	15′	45′	.5494	.9709	.5785	.4215	15′
10°	9.2397	9.9934	9.2463	0.7537	**80°**	**21°**	9.5543	9.9702	9.5842	0.4158	**69°**
15′	.2503	.9930	.2573	.7427	45′	15′	.5592	.9694	.5898	.4102	45′
30′	.2606	.9927	.2680	.7320	30′	30′	.5641	.9687	.5954	.4046	30′
45′	.2707	.9923	.2784	.7216	15′	45′	.5689	.9679	.6009	.3991	15′
	COS	SIN	COT	TAN	°		COS	SIN	COT	TAN	°

Logarithms of Trigonometric Ratios

°	SIN	COS	TAN	COT	
22°	9.5736	9.9672	9.6064	0.3936	**68°**
15′	.5782	.9664	.6118	.3882	45′
30′	.5828	.9656	.6172	.3828	30′
45′	.5874	.9648	.6226	.3774	15′
23°	9.5919	9.9640	9.6279	0.3721	**67°**
15′	.5963	.9632	.6331	.3669	45′
30′	.6007	.9624	.6383	.3617	30′
45′	.6050	.9616	.6435	.3565	15′
24°	9.6093	9.9607	9.6486	0.3514	**66°**
15′	.6135	.9599	.6537	.3463	45′
30′	.6177	.9590	.6587	.3413	30′
45′	.6219	.9582	.6637	.3363	15′
25°	9.6259	9.9573	9.6687	0.3313	**65°**
15′	.6300	.9564	.6736	.3264	45′
30′	.6340	.9555	.6785	.3215	30′
45′	.6379	.9546	.6834	.3166	15′
26°	9.6418	9.9537	9.6882	0.3118	**64°**
15′	.6457	.9527	.6930	.3070	45′
30′	.6495	.9518	.6977	.3023	30′
45′	.6533	.9508	.7025	.2975	15′
27°	9.6570	9.9499	9.7072	0.2928	**63°**
15′	.6607	.9489	.7118	.2882	45′
30′	.6644	.9479	.7165	.2835	30′
45′	.6680	.9469	.7211	.2789	15′
28°	9.6716	9.9459	9.7257	0.2743	**62°**
15′	.6752	.9449	.7302	.2698	45′
30′	.6787	.9439	.7348	.2652	30′
45′	.6821	.9429	.7393	.2607	15′
29°	9.6856	9.9418	9.7438	0.2562	**61°**
15′	.6890	.9408	.7482	.2518	45′
30′	.6923	.9397	.7526	.2474	30′
45′	.6957	.9386	.7571	.2429	15′
30°	9.6990	9.9375	9.7614	0.2386	**60°**
15′	.7022	.9364	.7658	.2342	45′
30′	.7055	.9353	.7701	.2299	30′
45′	.7087	.9342	.7745	.2255	15′
31°	9.7118	9.9331	9.7788	0.2212	**59°**
15′	.7150	.9319	.7831	.2169	45′
30′	.7181	.9308	.7873	.2127	30′
45′	.7212	.9296	.7916	.2084	15′
32°	9.7242	9.9284	9.7958	0.2042	**58°**
15′	.7272	.9272	.8000	.2000	45′
30′	.7302	.9260	.8042	.1958	30′
45′	.7332	.9248	.8084	.1916	15′
33°	9.7361	9.9236	9.8125	0.1875	**57°**
15′	.7390	.9224	.8167	.1833	45′
30′	.7419	.9211	.8208	.1792	30′
45′	.7447	.9198	.8249	.1751	15′
	COS	SIN	COT	TAN	°

°	SIN	COS	TAN	COT	
34°	9.7476	9.9186	9.8290	0.1710	**56°**
15′	.7504	.9173	.8331	.1669	45′
30′	.7531	.9160	.8371	.1629	30′
45′	.7559	.9147	.8412	.1588	15′
35°	9.7586	9.9134	9.8452	0.1548	**55°**
15′	.7613	.9120	.8493	.1507	45′
30′	.7640	.9107	.8533	.1467	30′
45′	.7666	.9093	.8573	.1427	15′
36°	9.7692	9.9080	9.8613	0.1387	**54°**
15′	.7718	.9066	.8652	.1348	45′
30′	.7744	.9052	.8692	.1308	30′
45′	.7769	.9038	.8732	.1268	15′
37°	9.7795	9.9023	9.8771	0.1229	**53°**
15′	.7820	.9009	.8811	.1189	45′
30′	.7844	.8995	.8850	.1150	30′
45′	.7869	.8980	.8889	.1111	15′
38°	9.7893	9.8965	9.8928	0.1072	**52°**
15′	.7918	.8950	.8967	.1033	45′
30′	.7941	.8935	.9006	.0994	30′
45′	.7965	.8920	.9045	.0955	15′
39°	9.7989	9.8905	9.9084	0.0916	**51°**
15′	.8012	.8890	.9122	.0878	45′
30′	.8035	.8874	.9161	.0839	30′
45′	.8058	.8858	.9200	.0800	15′
40°	9.8081	9.8843	9.9238	0.0762	**50°**
15′	.8103	.8827	.9277	.0723	45′
30′	.8125	.8810	.9315	.0685	30′
45′	.8148	.8794	.9353	.0647	15′
41°	9.8169	9.8778	9.9392	0.0608	**49°**
15′	.8191	.8761	.9430	.0570	45′
30′	.8213	.8745	.9468	.0532	30′
45′	.8234	.8728	.9506	.0494	15′
42°	9.8255	9.8711	9.9544	0.0456	**48°**
15′	.8276	.8694	.9582	.0418	45′
30′	.8297	.8676	.9621	.0379	30′
45′	.8317	.8659	.9659	.0341	15′
43°	9.8338	9.8641	9.9697	0.0303	**47°**
15′	.8358	.8624	.9735	.0265	45′
30′	.8378	.8606	.9772	.0228	30′
45′	.8398	.8588	.9810	.0190	15′
44°	9.8418	9.8569	9.9848	0.0152	**46°**
15′	.8437	.8551	.9886	.0114	45′
30′	.8457	.8532	.9924	.0076	30′
45′	.8476	.8514	.9962	.0038	15′
45°	9.8495	9.8495	0.0000	0.0000	**45°**
	COS	SIN	COT	TAN	°

7. From a point on a level with the base of a lighthouse a line to its top makes an angle of 21° with the horizontal. The distance of the point from the base of the lighthouse is 300 feet. How high is the lighthouse?

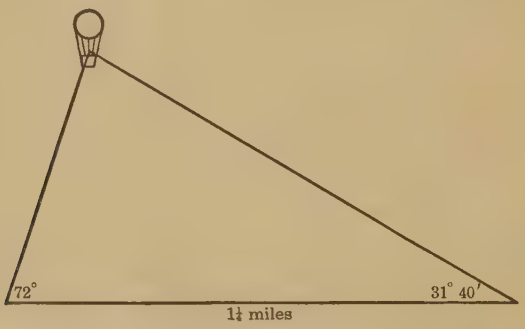


8. A line from a point A on the ground to the top of a flag pole makes an angle of 61° with the vertical. The distance to the flag pole from A is 320 feet. Find the height of the flag pole above A.

9. A tree casts a shadow 120 feet long when the sun is 42° 20′ above the horizon. Find the height of the tree.

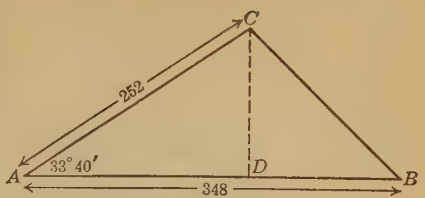
10. The length of a kite string is 1200 yards and the average angle it makes with the horizontal is 39°. How high is the kite?

11. Two observers facing each other on a level field noted the elevation of a balloon to be 72° and 31° 40′

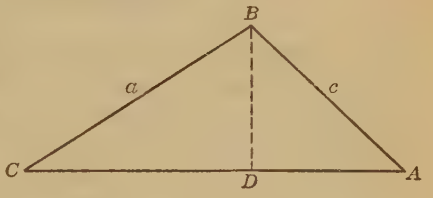


respectively. If the distance between the observers is $1\frac{1}{4}$ miles, find the height of the balloon.

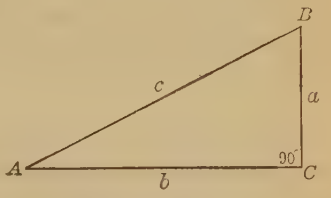
12. Two sides of a triangle are 348 and 252 and their included angle is 33° 40′. Find the altitude of the triangle and its area.



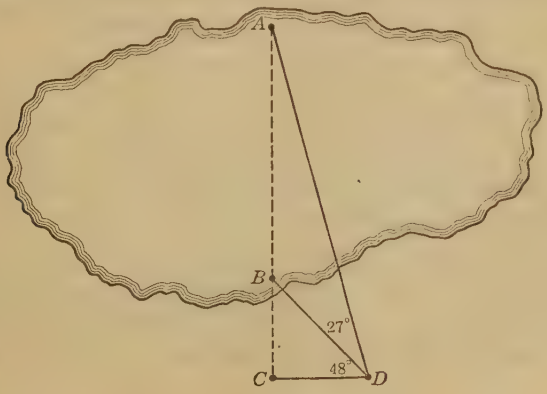
13. Show that in any triangle the altitude $BD = a \sin C$ and the area $= \frac{ab \sin C}{2}$ or $\frac{bc \sin A}{2}$.



14. Show that the area of a right triangle is $\frac{1}{2}\, bc \sin A$, $\frac{1}{2}\, b^2 \tan A$, or $\frac{1}{2}\, a^2 \tan B$.



15. A and B are two channel markers in White Lake. C is a point on the shore, in the line of AB, and CD, 1000 feet long, is perpendicular to AC. The angle BDC is 48° and $CDA = 75°$. Find the distance AB.



CHAPTER XIX

PROGRESSIONS

135. A sequence of numbers. In all fields of mathematics we frequently encounter groups of three or more numbers, selected according to some law and arranged in a definite order, whose relations to each other and to other numbers we wish to study.

There is an unlimited variety of such groups, or successions, of numbers. Only two simple types will be considered here.

136. Arithmetic progression. An *arithmetic progression* is a succession of terms in which each term after the first is formed by adding the same number to each preceding number.

Thus, if a denotes the first term and d the common number added, any arithmetic progression is represented by

$$a,\ a + d,\ a + 2\,d,\ a + 3\,d,\ a + 4\,d,\ \cdots.$$

This common number d is called the common difference and may be any number, *positive* or *negative*. It may be found for any given arithmetic progression by subtracting any term from the term which follows it.

The numbers 4, 7, 10, 13, $\cdots$ form an arithmetic progression, since any term, after the first, minus the preceding one gives 3. Similarly, 11, 7, 3, -1, -5, $\cdots$ is an arithmetic progression, since any term, after the first, minus the preceding one

gives the common difference -4. In like manner $\frac{7}{3}$, 4, $5\frac{2}{3}$, $\cdots$ is an arithmetic progression whose common difference is $1\frac{2}{3}$.

ORAL EXERCISES

State the first four terms of the arithmetic progressions:

1. $a = 3$, $d = 4$.
2. $a = 10$, $d = 7$.
3. $a = 4$, $d = 5$.
4. $a = 15$, $d = 6$.
5. $a = 40$, $d = -1$.
6. $a = 80$, $d = -4$.
7. $a = -5$, $d = -2$.
8. $a = 40\,m$, $d = -2\,m$.
9. $a = m$, $d = 2\,m - 1$.
10. $a = -5$, $d = -2$.

Which of the following are arithmetic progressions:

11. 1, 7, 12.
12. 3, 5, 7.
13. 20, 16, 11.
14. 8, 6, 4.
15. $3\,m$, $4\,m - 2$, $5\,m - 4$.
16. $6\frac{3}{8}$, $9\frac{5}{8}$, $11\frac{7}{8}$.
17. 14, 3, -8.
18. $5\,m$, $4\,m + 2$, $3\,m + 4$.
19. 2, 4, 8.
20. $2\,m - 1$, m, 1, $2 - x$.

21. Give a verbal definition of an arithmetic progression.

22. Give a symbolic definition of an arithmetic progression.

137. The last or *n*th term of an arithmetic progression. In the symbolic definition of an arithmetic progression

$$a,\ a + d,\ a + 2\,d,\ a + 3\,d,\ a + 4\,d,\ \cdots$$

the coefficient of d in the *second* term is 1, in the *third* term it is 2, in the *fourth* term it is 3, etc.; that is, in each term it is one less than the number of the term. Hence, in the nth, or general, term, the coefficient of d is $n - 1$. Hence, if l denotes the nth term, we have

$$l = a + (n - 1)d.$$

EXERCISES

Find the required terms in the following arithmetic progressions:

1. The tenth term of 2, 5, and 8.

Solution. Here $a = 2$, $d = 3$ and $n = 10$.

Hence $l = a + (n - 1)d$

becomes $l = 2 + (10 - 1)3 = 29$.

2. The ninth term of 3, 7, 11, ···.

3. The fourteenth term of 8, 6, 4, ···.

4. The twentieth term of m, $5\,m$, $9\,m$, ···.

5. The thirtieth term of -10, $-8\frac{1}{3}$, $-6\frac{2}{3}$, ···.

6. The fifteenth term of $\frac{4}{5}$, $\frac{2}{5}$, 0, ···.

7. The twentieth term of m, $2\,m + 1$, $3\,m + 2$, ···.

8. The nth term of 2, 4, 6, ···.

9. Determine a general form for any even number.

10. The nth term of 1, 3, 5, ···.

11. Determine a general form for any odd number.

12. The nth term of 7, 9, 11, ···.

13. n − 1st term of the series in Exercises 8, 10, 12.

14. The nth term of the series 8, 10, 12, ···.

15. The nth term of $\frac{1}{c}$, $\frac{a-1}{c}$, $\frac{2\,a-3}{c}$.

16. Find the amount of seventy-five dollars at six per cent simple interest for four years.

17. Find the amount of P dollars for n years at r per cent simple interest.

18. What does the expression in Problem 17 become when $n = 1$? 2? 3? Are these three results in arithmetic progression?

19. A body falls 16 feet the first second, 48 feet the next, 80 feet the next, and so on. How far does it fall during the tenth second? the twentieth? the nth?

20. The digits of a certain three-digit number are in arithmetic progression. The sum of the digits is 15 and the sum of their squares is 93. Find the number.

21. If an arithmetic progression has an odd number of terms, show that the middle term is half the sum of the first and last terms.

22. In an arithmetic progression find the 9th term from the beginning; the 9th term from the end.

23. The pairs of numbers $(1, \frac{3}{4})$, $(2, \frac{3}{2})$, $(3, \frac{9}{4})$, $(4, 3)$, $(5, \frac{15}{4})$, $(6, -\frac{9}{2})$ are respectively the x and y of coördinates of six points. Show that the distances of the points from the origin are in arithmetic progression.

24. Find the rth term from the beginning of an arithmetic progression and the rth term from the end. Show that the arithmetic average of these two terms is the middle term of the progression if it has an odd number of terms.

25. The velocity of a body falling from rest increases uniformly 32 feet per second each second. Find the velocity at the end of the 10th second. Find the average velocity during the first second, during the third second, during the nth second.

138. Arithmetic means. The arithmetic mean between two numbers is a number which forms with the two given ones as the first and last terms an arithmetical progression.

One important special case of arithmetic means is the arithmetic *mean between two numbers* h and k. This is $\frac{h+k}{2}$ or merely the average of the two numbers.

Proof. There would be three terms in the progression, h being the first and k the last.

Then $l = a + (n-1)d$

becomes $k = h + (3-1)d.$

Hence $d = \frac{k-h}{2}.$

Therefore the series is $h,\ h + \left(\frac{k-h}{2}\right),\ k$

or $$h,\ \frac{h+k}{2},\ k.$$

EXERCISES

1. Insert five arithmetic means between 8 and 92.

Solution. There will be seven terms in all.

Therefore $l = a + (n-1)d$

becomes $92 = 8 + (7-1)d = 8 + 6d.$

Hence $d = 14.$

Therefore the series is 8, 22, 36, 50, 64, 78, 92.

2. Insert 4 arithmetic means between 100 and 200.
3. Insert 10 arithmetic means between 11 and 99.
4. Insert 6 arithmetic means between 70 and 800.

Find the arithmetic mean between:

5. 18, 60.
6. 30, 400.
7. a, c.
8. x, $x - 2y$.
9. x, $-x$.
10. $a + b$ and $a - b$.

11. What is the arithmetic mean of the odd numbers from 23 to 37 inclusive?

12. An automobile starting from rest increased its speed in 18 seconds to 66 feet per second. What was its final rate in miles per hour? What was its average rate in miles per hour during the time its speed was increasing?

139. The sum of a series. The indicated sum of several terms of an arithmetic progression is called an arithmetic series. The formula for the sum of n terms of an arithmetic series is derived as follows:

$$S_n = a + (a + d) + (a + 2d) + \cdots [a + (n - 3)d] + [a + (n - 2)d] + [a + (n - 1)d] \quad (1)$$

Writing (1) again and reversing the order of the terms in the right member gives:

$$S_n = [a + (n - 1)d] + [a + (n - 2)d] + [a + (n - 3)d] + \cdots (a + 2d) + (a + d) + a. \quad (2)$$

Adding (1) and (2) gives

$$2S_n = [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)d] + \cdots [2a + (n - 1)d] + [2a + (n - 1)d] + [2a + (n - 1)d] \quad (3)$$

Since there are n terms in the right members of (1) and (2), there are n brackets in (3); therefore

$2S_n = n[2a + (n - 1)d]$ or

$$S_n = \frac{n}{2}[2a + (n - 1)d]$$

EXERCISES

Find the sum in the following arithmetic progressions:

1. The first 18 terms of 3, 5, 7.

Solution. Here $a = 3$, $d = 2$, and $n = 18$.

Hence $s = \frac{n}{2}[2a + (n-1)d]$

becomes $s = \frac{18}{2}[2 \cdot 3 + (18-1)2] = 9(6+34) = 360.$

2. The first 50 terms of $1 + 2 + 3 + \cdots$.

3. The first 8 terms of $60 + 57 + 54 + \cdots$.

4. The first 10 terms of $1 + 1\frac{1}{2} + 2 + \cdots$.

5. All the even numbers less than 101.

6. All the odd numbers between 100 and 200.

7. The sum of the first ten thousand integers.

8. The sum of the 20 terms of $2 + 2.2 + 2.4 + \cdots$.

9. Find the sum of the integers between 200 and 600 which are exact multiples of 7.

10. Find the sum of the numbers less than 100 which are exact multiples of 11.

11. The sum of 15 terms of $(3 + \sqrt{2}) + 3 + (3 - \sqrt{2}) + \cdots$.

12. The sum of 10 terms of $\frac{1}{c} + \frac{c-1}{c} + \frac{2c-3}{c} + \cdots$.

13. The sum of n terms of $(a - c) + a + (a + c) + \cdots$.

14. The sum of n terms of $3c + c - c \cdots$.

15. The sum of n terms of $4\frac{1}{2} + 3\frac{1}{2} + 2\frac{1}{2} + \cdots$.

16. The sum of n terms of $1 + 3 + 5 + \cdots$.

17. The sum of n terms of $2 + 4 + 6 + \cdots$.

18. Find the sum of the first n integers which are exact multiples of 13.

19. How many terms of $1 + 5 + 9 + \cdots$ will amount to 630?

HINT. We have here S, a, and d in $S = \frac{n}{2}[2\,a + (n-1)d]$, to solve the equation for n.

20. How many terms of $15 + 12\frac{1}{2} + 10 + \cdots$ will amount to 15?

21. How many terms of $-12 - 10\frac{1}{2} - 9 \cdots$ will amount to -54?

22. How many terms of $16 + 15\frac{2}{3} + 15\frac{1}{3} + \cdots$ will amount to 216?

23. The first term of an arithmetic progression is 7 and the 6th term is 12. Find the sum of the first 100 terms.

24. Assuming that a heavy iron ball is not retarded by the air, determine the number of seconds it would take to fall if it is dropped from a point on the Eiffel Tower 980 feet above the ground. ($S = 16\,t^2$.)

25. A ball shot vertically upward returned to the point it started from 18 seconds later. How high did it rise?

26. By Exercise 25, page 277, it is seen that a falling body obeys the law of an arithmetic progression. Show from the data of that exercise that the general formula

$$S = \frac{n}{2}[2\,a + (n-1)d]$$

becomes the special one, $S = \frac{gt^2}{2}$ used in physics for such problems.

27. A passenger in an aëroplane drops a rock and observes that it strikes the ground 11 seconds later. How high was the plane at the time?

28. A man deposits in a bank 18 dollars on the first of each month from January 1, 1924, to December 1, 1925. What is the amount of his deposit at that time?

29. The bank in Exercise 28 gives three per cent interest, paid monthly, on deposits. What will the interest amount to by January 1, 1925?

30. A man buys a piano for $900 by paying $200 cash and the remainder in monthly instalments of $100 and interest at six per cent on the unpaid balance. What was his total payment?

HINT. The monthly balances are respectively $700, $600, $500, etc. The corresponding interest payments are $3.50, $3.00, $2.50, etc.

31. A man bought a house for $11,500, paying $1500 cash and giving a first mortgage for $6000 and a second mortgage for $4000, each bearing 6% interest. Payments of $200 plus the accrued interest were made every four months on the second mortgage. Find the sum of all interest payments on the second mortgage.

32. Two men 2010 feet apart start at the same time and move toward each other. One moves 8 feet the first second, 13 feet the second, and 18 feet the third, etc. The other moves 7 feet the first second, 11 the second, 15 the third, and so on. In how many seconds will they meet?

33. A pyramid of balls all of the same size stands on an equilateral triangle, twelve balls on a side. How many balls are there in each layer? in the whole pyramid?

34. A and B set out from the same place at the same time and travel in the same direction. A travels 20 miles per hour. B goes 32 miles the first hour, 28 the second, 24 the third, and so on. When will they be together again?

35. A man leaves a certain place and travels 16 miles the first hour, 18 the second, 20 the third, and so on. One hour and twenty minutes later a second man leaves the same place and travels the same road at a constant rate of 30 miles per hour. When are the two together?

36. The successive intervals between a number of stones placed in a straight line on the ground is 1 rod, 3 rods, 5 rods, etc. A boy collects the stones one by one and places them in a box located at the first stone. How far will he travel in handling the 7th stone?

37. In Exercise 36 determine how far the boy will travel for the nth stone; the $n+1$st stone; the $n-1$st stone.

38. From the results of Exercise 37 determine the series which represents the total distance the boy must travel in collecting the first n stones. Can this series be summed by any formula developed thus far?

NOTE. In the earliest mathematical work known a problem is found which involves the idea of an arithmetic progression. In the papyrus of the Egyptian priest Ahmes, who lived nearly two thousand years before Christ, we read in essence, "Divide 40 loaves among 5 persons so that the number of loaves that they receive shall form an arithmetic progression, and so that the two who receive the least bread shall together have one seventh as much as the others." From that time to this, the subject has been a favorite one with mathematical writers, and has been extended so widely that it would require many volumes to record all of the discoveries regarding the various kinds of series.

140. Geometric progression. A *geometric progression* is a succession of terms in which each term after the first is obtained from the preceding one by multiplying it by the same number.

If a denotes the first term and r the common multiplier, the progression is represented by $a, ar, ar^2, ar^3, ar^4, \cdots$.

The common multiplier is called the *ratio*. It is evident from the definition that the ratio is obtained by dividing any term except the first by the preceding one.

The numbers 2, 6, 18, 54, ··· form a geometrical progression since any term after the first divided by the preceding one gives the same number 3. Similarly, the numbers 4, $-4\sqrt{5}$, 20, $-20\sqrt{5}$, ··· since from the second term on each term divided by the preceding one gives the common ratio $-\sqrt{5}$.

ORAL EXERCISES

Determine which of the following are geometric progressions and the ratio:

1. 1, 4, 8, 16, ···.
2. 3, 9, 27, ···.
3. 2, 4, 6, 8, ···.
4. 2, 1, $\frac{1}{2}$, ···.
5. 6, 2, $\frac{1}{3}$, ···.
6. 3, $\frac{1}{3}\sqrt{3}$, $\frac{1}{9}\sqrt{3}$, ···.
7. $\frac{1}{\sqrt{8^3}}$, $\frac{\sqrt{2}}{2}$, 2, ···.
8. $5a^2$, $10a^2x$, $20a^2x^2$, ···.
9. 10, 6, $\frac{18}{5}$, ···.
10. $ax^{\frac{1}{2}}$, $4a^{\frac{1}{2}}$, $16x^{\frac{3}{2}}$, ···.
11. Find the condition under which a, b, and c form a geometric progression.

State in order the first four terms of a geometric progression in which the ratio r and the first term a are given:

12. $a = 5$, and $r = 2$.
13. $a = 7$, and $r = 3$.
14. $a = 10$, and $r = -2$.
15. $a = 6$, and $r = \frac{1}{3}$.
16. $a = \frac{4}{5}$, and $r = -\frac{1}{2}$.
17. $a = 7$, and $r = \sqrt{2}$.

141. The *n*th term of a geometrical progression. In the symbolic definition of a geometrical progression

$$a,\ ar,\ ar^2,\ ar^3,\ ar^4,\ \cdots$$

the exponent of r in the second term is 1, in the third term 2, in the fourth term 3, etc.; that is, in each term it is one less than the number of the term. Therefore the nth or general term is

$$t_n = ar^{n-1}$$

EXERCISES

1. Find the sixth term of 3, 6, 12, ···.

Solution. Here $a = 3$, $r = 2$, and $n = 6$.

Therefore $t_n = ar^{n-1}$

becomes $t_n = 3 \cdot 2^5 = 3 \cdot 32 = 96$.

2. Find the seventh term of 2, 4, 8, ···.

3. Find the eighth term of 1, $\frac{1}{3}$, $\frac{1}{9}$, ···.

4. Find the tenth term of 5, -10, 20, ···.

5. Find the t_9 of 12, -4, $\frac{4}{3}$, ···.

6. Find the t_{10} of $24a^2$, $18a^4$, $\frac{27a^6}{2}$, ···.

7. Find t_8 of $9\sqrt{3}$, 9, $3\sqrt{3}$, ···.

8. Find t_7 of $\frac{1}{4}\sqrt{2}$, $\frac{1}{2}$, $\frac{1}{2}\sqrt{2}$, ···.

9. The number .23232323 is the sum of four terms in geometric progression. What are these terms? ($r = \frac{1}{100}$.)

10. The amount, A, of P dollars at r per cent interest compounded annually for n years is expressed by the formula $A = P(1 + r)^n$. Derive this formula.

11. Find the amount of \$500 at four per cent interest compounded annually for 4 years.

12. The first two terms of a geometrical progression are a and b. Find the third term.

13. Find the amount of 100 dollars at $5\frac{1}{2}\%$ compounded annually for 3 years.

14. What is the $(n-1)$st term of a geometric progression? the $(n-2)$d? the $(n-3)$d? the $(n+1)$st?

15. The sum of the digits of a three-digit number is 14 and the digits are in geometric progression. If 594 is added to the number, the result is expressed by the digits in reverse order. Find the number.

16. Show that the distances of the following points from the origin are in geometric progression: $(1, 3)$, $(\sqrt{2}, 3\sqrt{2})$, $(2, 6)$, $(2\sqrt{2}, 6\sqrt{2})$.

17. A house worth \$8000 depreciates 10% in value each year. What will it be worth at the end of 7 years?

18. An automobile selling at \$800 depreciates in use yearly 30% of its cost. What is it worth at the end of 3 years?

19. Show that the product of any two terms of a geometric progression equidistant from a given term is always the same.

20. Show that if each term of a geometric progression is subtracted from the following one, the successive remainders form a geometric progression.

21. One hundred dollars is deposited annually for six years in a bank which pays four per cent compound interest. What sum is due the depositor at the end of six years if he has withdrawn no money from the bank during that time?

22. A father deposits a certain sum in a bank at the beginning of the school year each of the four years his son is in the high school. If this amounts to sixteen hundred dollars when the boy enters college after his high school

course, find the yearly deposit if interest is compounded annually at four per cent.

23. A good ear of Dent corn has about 400 grains. Five per cent of the grains on such an ear are unfit to plant. If the others are planted and each produces an ear like the first, how many bushels of corn will there be in the fifth year's crop if each ear weighs 15 ounces and there are 70 pounds of corn in a bushel?

142. Geometric means. The geometric means between any two numbers are numbers which form, with the given ones as the first and last terms, a geometric progression.

One important special case is the *geometric mean* between two numbers h and k. This is $\pm\sqrt{h \cdot k}$.

Proof. There would be three terms in the progression, h being the first and k the last.

Then $t_n = ar^{n-1}$

becomes $k = hr^2$.

Hence $r = \pm\sqrt{\frac{k}{h}}$.

$\therefore$ the second term is $h\left(\pm\sqrt{\frac{k}{h}}\right) = \pm\sqrt{\frac{k \cdot h^2}{h}} = \pm\sqrt{kh}$.

Hence the series is $h, \pm\sqrt{hk}, k$.

EXERCISES

1. Insert three geometric means between 3 and 48.

Solution. The number of terms is five.

Therefore $48 = 3 \cdot r^4$.

Hence $r^4 = 16$ and $r = \pm 2$.

$\therefore$ the series is $3, \pm 6, 12, \pm 24, 48$.

2. Insert four geometric means between 1 and 243.

3. Insert two geometric means between 19 and 152.

4. Insert three geometric means between -7 and -112.

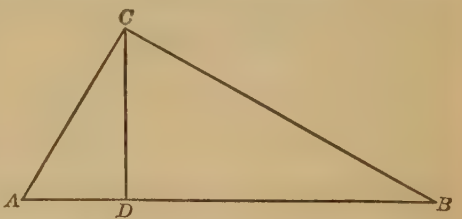
5. Insert one geometric mean between b and c.

6. Insert two geometric means between b and c.

7. The second and third terms of a geometric progression are a and b. Find the first term.

8. The arithmetic mean of two numbers is 26 and their geometric mean is 24. Find the numbers.

9. In the right triangle CD is perpendicular to the hypotenuse AB. It is proved in geometry that CD is a geometric mean between AD and BD.

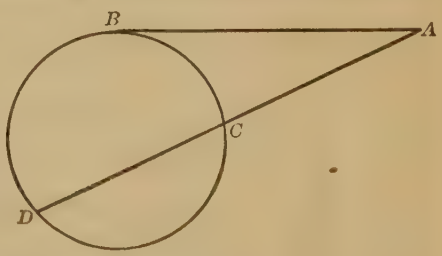


(a) If $AD = 9$ and $BD = 16$, find CD.

(b) If $CD = 6$ and $BC = 13$, find AD and BD.

(c) If $DC = 20$ and $AD = 15$, find BD, AC, and BC.

10. In the adjacent figure AB touches and AD intersects the circle. It is proved in geometry that AB is a geometric mean between AC and AD.



(a) If $AB = 10$ and AD is 90, find AC.

(b) If $DC = 7$ and $AB = 12$, find AC and AD.

143. Geometric series. The indicated sum of n terms of a geometric progression is called a *geometric series.*

In symbols this sum, S_n, is indicated by

$$S_n = a + ar + ar^2 + ar^3 + \cdots ar^{n-1}.$$

In practice the simplest form of this indicated sum is needed. The value of S_n in the simplest form is derived as follows:

$$S_n = a + ar + ar^2 + ar^3 + \cdots ar^{n-3} + ar^{n-2} + ar^{n-1}. \quad (1)$$

Multiplying both members of (1) by (r) gives

$$S_n r = ar + ar^2 + ar^3 + ar^4 \cdots ar^{n-2} + ar^{n-1} + ar^n. \quad (2)$$

Subtracting (2) from (1), all terms on the right vanish except a and ar^n, giving $S_n - S_n r = a - ar^n. \quad (3)$

$$S_n(1 - r) = a - ar^n. \quad (4)$$

Solving (4), $$S_n = \frac{a - ar^n}{1 - r}.$$

EXERCISES

1. Find the sum of the first eight terms of $2 + 6 + 18 + \cdots$.

Solution. Here $a = 2$, $r = 3$, and $n = 8$.

Substituting in $S_n = \dfrac{a - ar^n}{1 - r}$ gives

$$S_8 = \frac{2 - 2 \cdot 3^8}{1 - 3} = \frac{2 - 2 \cdot 3^8}{-2}$$

$$= -1 + 3^8 = -1 + 6561 = 6560.$$

2. Find the sum of nine terms of $3 + 6 + 12 + \cdots$.

3. Find the sum of seven terms of $5 - 10 + 20 - \cdots$.

4. Find the sum of eight terms of $80 + 40 + 20 + \cdots$.

5. Find the sum of ten terms of $1 + \frac{1}{2} + \frac{1}{4} + \cdots$.

6. Find S_7 for $2\sqrt{2} - 4 + 4\sqrt{2} + \cdots$.

7. Find S_9 for $3 + \frac{3}{2} + \frac{3}{4} + \cdots$.

8. Find S_{10} for $\frac{3}{4} + 1 + \frac{4}{3} + \cdots$.

9. Find S_9 for $x^2 + x^4 + x^6 + \cdots$.

10. Find S_8 for $m + 2m^3 + 4m^5 + \cdots$.

11. Find the sum of n terms of $5 + 10 + 20 + \cdots$.

12. Find the sum of $n - 1$ terms of $c + c^4 + c^7 + \cdots$.

13. Find the sum of seven terms of $\frac{a}{c} + \frac{c}{a} + \frac{c^3}{a^3} + \cdots$.

14. Find the sum of $n - 3$ terms of $\frac{a}{c} + 1 + \frac{c}{a} + \cdots$.

15. The sum of the first three terms of a geometric progression is 65 and the difference of the first and third is 40. Find the terms.

16. The sum of the first four terms of a geometric progression is 30 and the sum of the next four is 480. Find the series.

17. The sum of three numbers which are in arithmetic progression is 21. If 1, 5, and 25 be added to them respectively, the results are in geometric progression. Find the numbers.

18. A rubber ball falls to the floor from a height of 72 inches. At each rebound it rises thirty per cent of its previous height. How far has it moved when it strikes the floor the tenth time?

19. A vessel containing alcohol was emptied of one fourth of its contents and then filled up with water. This

was done five times. What portion of the original contents was then in the vessel?

20. At each stroke an air pump withdraws 20 cubic inches of air from a vessel containing 200 cubic inches. After every stroke the air remaining in the vessel expands and completely fills it. What fraction of the original quantity of air remains in the vessel at the end of the eighth stroke?

144. Infinite geometric series. If the number of terms of a geometric series is unlimited, it is called an infinite geometric series.

In the progression 3, 6, 12, ··· the ratio is positive and greater than 1 and each term is greater than the preceding one. Such a progression is an *increasing* one. It is obvious that the sum of an unlimited number of terms in an increasing geometric progression is unlimited. If $r > 1$ (read "r is greater than 1"), the sum can be made as large as we please by taking a sufficient number of terms.

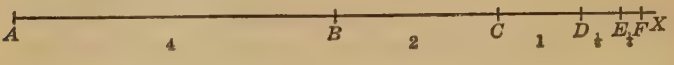
If the geometric progression is a decreasing one, however, the result is very different. In the decreasing progression 4, 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, ··· the ratio is positive and less than 1, and each term is less than the preceding one. Here the sum of the first three terms is 7, the first four $7\frac{1}{2}$, the first five $7\frac{3}{4}$, the first six $7\frac{7}{8}$, etc. Obviously, the more terms taken the greater is the sum. But no matter how large the number of terms taken their sum is less than 8. This number 8 is called the *limit* of the series.

$$S_n = 4 + 2 + 1 + \tfrac{1}{2} + \tfrac{1}{4} \cdots.$$

This series and its limit can be illustrated by the motion of a point along a line in the diagram on the following page.

Suppose line AX is eight inches long and a point starts from A and moves in one minute to point B which is $\frac{1}{2}$ the

distance to X. In the next minute it moves to C, $\frac{1}{2}$ the remaining distance. In the third minute it moves to D, $\frac{1}{2}$ the remaining distance, etc. But the point will never reach X nor will the total distance it moves ever quite equal eight inches.



The formula $S_n = \dfrac{a - ar^n}{1 - r}$

becomes $$S_n = \frac{a}{1 - r} - \frac{ar^n}{1 - r}. \tag{1}$$

For the preceding series we have $a = 4$, $r = \frac{1}{2}$, and $n = n$.

Substituting, $$S_n = \frac{4}{1 - \frac{1}{2}} - \frac{4(\frac{1}{2})^n}{1 - \frac{1}{2}}. \tag{2}$$

$$S_n = \frac{4}{\frac{1}{2}} - \frac{4(\frac{1}{2})^n}{\frac{1}{2}} = 8 - 4(\tfrac{1}{2})^{n-1}. \tag{3}$$

Now $(\frac{1}{2})^2 = \frac{1}{4}$, $(\frac{1}{2})^3 = \frac{1}{8}$, $(\frac{1}{2})^4 = \frac{1}{16}$, etc. Hence $(\frac{1}{2})^n$ becomes very small if n is great. Consequently $4(\frac{1}{2})^{n-1}$ approaches *zero* when n increases *without limit*. Therefore the right member of (3) approaches 8 if n increases without limit. That is, the limit of S_n is $\dfrac{4}{\frac{1}{2}}$ or 8. In general, then, when we speak of the *sum* of an infinite geometric series we mean the *limit* which the sum approaches as n increases indefinitely.

Moreover, the formula (1) above becomes for a decreasing series

$$S_\infty = \frac{a}{1 - r}.$$

EXERCISES

Find the sum of the following infinite geometric series:

1. $3 + 1 + \frac{1}{3} + \cdots$.

Solution. $s = \dfrac{a}{1 - r}$.

Substituting, $s = \dfrac{3}{1 - \frac{1}{3}} = 4\frac{1}{2}$.

2. $8 + 4 + 2 + 1 + \cdots$.

3. $9 + 3 + 1 + \cdots$.

4. $12 + 3 + \frac{3}{4} + \cdots$.

5. $25 + 5 + 1 + \cdots$.

6. $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$.

7. $2 + \sqrt{2} + 1 + \cdots$.

8. $3 - \sqrt{3} + 1 + \cdots$.

9. $1 + x + x^2 + \cdots$ where $x < 1$.

10. $6 + \sqrt{6} + 1 + \cdots$.

11. $1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \cdots$ where $x > 1$.

12. $(m + n) + 1 + \dfrac{1}{m + n} + \cdots$.

13. $.323232 \cdots$.

HINT. This equals

$$\tfrac{32}{100} + \tfrac{32}{10{,}000} + \tfrac{32}{1{,}000{,}000} + \cdots$$

14. $.777 \cdots$.

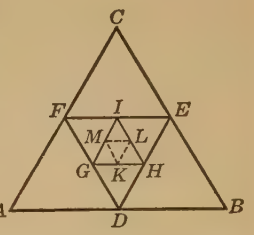
15. $.6464 \cdots$.

16. $.020202 \cdots$.

17. $.714285714285 \cdots$.

18. $3.4242 \cdots$.

19. *ABC* is an equilateral triangle with side *AB* equal to 4. The triangle *DEF* is formed by joining the midpoints of the adjacent sides of *ABC*. Triangle *IGH* is formed similarly, and so on. Determine the series formed by the areas of the successive triangles. Find the sum of the areas of all the triangles drawn as there supposed.



20. A flywheel whose circumference is 12 feet makes 2400 revolutions per minute. If it slows down so that each second it makes ninety-eight per cent as many revolutions as it did the preceding second, how far will a point on its rim have traveled by the time it has stopped?

NOTE. In the study of geometric progressions we have seen that the sum of the infinite series $1 + x + x^2 + x^3 + \cdots$ is a definite number when x has any value less than 1. But it has no finite value when x is equal to or greater than 1; that is, we have an expression which we cannot use arithmetically unless x has a properly chosen value. If we were studying a problem which involved such a series, it would be a matter of the most vital importance to know whether the values of x under discussion were such as to make the series meaningless.

This question of distinguishing between expressions which converge (that is, the sum of whose terms approaches a limit) and those which do not has an interesting history. Newton and his followers in the seventeenth century dealt with infinite series and always assumed that they converged, as, in fact, most of them did. But as more complicated series came into use it became more difficult to tell from inspection whether the series converged or not for a given value of the variable.

It was not until the beginning of the nineteenth century that Gauss, Abel, and Cauchy, in Germany, Norway, and France respectively, began to teach this subject effectively and to devise far-reaching tests to determine the values of x for which certain series converge to a finite limit. It is said that on hearing a discussion by Cauchy in regard to series which do not always converge, the astronomer Laplace became greatly alarmed lest he had made use of some such series in

his great work, "Celestial Mechanics." He hurried home and denied himself to all distractions until he had examined every series in his book. To his intense satisfaction they all converged.

MISCELLANEOUS EXERCISES

1. Define an arithmetic progression:

(*a*) in symbols, (*b*) in words.

2. Derive the formula for the nth term of an arithmetic progression.

3. Derive the formula for the sum of n terms of an arithmetic progression.

4. What is the arithmetic mean between two numbers?

5. Define a geometric progression:

(*a*) in symbols, (*b*) in words.

6. Derive the formula for the nth term of a geometric progression.

7. Derive the formula for the sum of n terms of a geometric progression.

8. Show that this last formula is equivalent to

$$S_n = \frac{a - rl}{1 - r}.$$

9. Show that for a converging geometric progression

$$S_n = \frac{a}{1 - r}.$$

10. In the equation

$$S(1 + r)^4 - p(1 + r)^3 - p(1 + r)^2 - p(1 + r) - p = 0$$

show that $$p = \frac{Sr(1 + r)^4}{(1 + r)^4 - 1}.$$

11. A loan of S dollars is to be repaid in four equal annual payments of p dollars each. Find p if money is worth r %.

Solution. The sum due at beginning of second year is

$$S(1 + r) - p. \tag{1}$$

The sum due at beginning of third year is

$$[S(1 + r) - p](1 + r) - p. \tag{2}$$

The sum due at beginning of fourth year is

$$\{[S(1 + r) - p](1 + r) - p\}(1 + r) - p. \tag{3}$$

The sum due at beginning of fifth year is

$$[\{[S(1 + r) - p](1 + r) - p\}(1 + r) - p](1 + r) - p. \tag{4}$$

By the conditions of the problem, (4) = 0, for all the debt has then been paid. Setting (4) equal to zero and simplifying,

$$S(1 + r)^4 - p(1 + r)^3 - p(1 + r)^2 - p(1 + r) - p = 0. \tag{5}$$

Solving (5) for p,

$$p = \frac{S(1 + r)^4}{(1 + r)^3 + (1 + r)^2 + (1 + r) + 1}. \tag{6}$$

But the denominator in (6) is a geometrical series whose sum by the formula $S_n = \frac{a - ar^n}{1 - r}$ is $\frac{(1 + r)^4 - 1}{r}$.

Substituting this last value for the denominator of (6),

$$p = \frac{Sr(1 + r)^4}{(1 + r)^4 - 1}. \tag{7}$$

In the general case, if we have n annual payments, the exponents 4 in (7) would be replaced by n, and then

$$p = \frac{Sr(1 + r)^n}{(1 + r)^n - 1}.$$

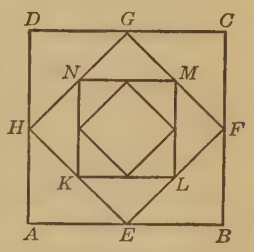
12. A loan of \$600 is to be repaid in four equal annual payments, with interest at 6%. Find the amount of each payment.

13. A loan of \$1000 is to be repaid in four equal annual payments with interest at 5%. Find the payment required.

14. The equipment of a certain factory cost \$25,000. The estimated depreciation in its value is 8% yearly. On that basis compute its value at the end of 5 years.

15. One third the contents of a vessel containing acid was taken out and replaced by water. This was done six times. What portion of the acid remained in the vessel?

16. Each square in the adjacent figure except the first is formed by joining the mid-points of the square next larger. If $AB = 8$, show that the perimeters of the squares form a decreasing geometrical progression. Find the sum of the perimeters of all the squares which may be so drawn.



CHAPTER XX

RATIO, PROPORTION, AND VARIATION

145. Ratio. The ratio of one number a to a second number b is the quotient obtained by dividing the first by the second, or $\frac{a}{b}$. The ratio of a to b is also written $a : b$.

It follows from the above that all ratios of two numbers are fractions and all fractions may be regarded as ratios.

Thus $\frac{3}{4}$, $\frac{a}{2c}$, $\frac{m+3}{m-n}$, and $\frac{\sqrt{3}}{\sqrt{5}}$ are ratios as well as fractions.

Since ratios like the above are fractions, operations which may be performed on fractions may be performed on these ratios. Hence the value of a ratio is not changed by multiplying or dividing both numerator (antecedent) and denominator (consequent) by the same number.

EXERCISES

Simplify the following ratios by considering them as fractions and reducing the fractions to lowest terms:

1. 15 inches : 1 rod.
2. 100 inches : $\frac{3}{10}$ mile.
3. 2 pints : 7 gallons.
4. 1 mile : 1 kilometer (1 meter = 39.37 inches).
5. .02 ton : 1 kilogram (1 kilogram = 2.205 lbs.).
6. A lot 100 × 135 feet : 1 acre.

7. Which is the greater ratio 17 : 25 or 23 : 33?

8. In an alloy of 52 ounces of zinc and copper there are 12 ounces of zinc. Find the ratio of zinc to copper.

9. Divide \$1800 into two parts which are in the ratio of 5 : 7.

10. Separate 360 into three parts in the ratio of 3 : 5 : 7.

11. Show that $\frac{x}{x+3} < \frac{x+4}{x+7}$ if x is positive.

HINT. Reduce the fractions to respectively equivalent fractions having a common denominator and then compare the numerators.

12. Show $\frac{x}{x+5} < \frac{x+3}{x+8}$ if x is positive.

13. Show $\frac{x}{x+3} < \frac{x+4}{x+5}$ if x is positive.

14. Arrange the ratios 3 : 4; 7 : 8; 13 : 16 in decreasing order of magnitude.

15. Simplify $\left(1 - \frac{9}{x^4}\right) : \left(1 + \frac{3}{x^2}\right)$.

16. $\frac{2a^2 - 5a + 3}{a^2 - 9} : \frac{a^2 - 2a + 1}{a^2 - 5a + 6}$.

17. $\frac{a(x^2 - 3a) - ax(2x - 3)}{(x^2 - 3x)^2} : \frac{ax}{x^2 - 3x}$.

18. Which is the greater ratio if a is positive:

$$\frac{7 + 2a}{7 + 3a} \text{ or } \frac{7 + 4a}{7 + 5a}?$$

19. If a positive number is added to or subtracted from both terms of a proper fraction, what change is produced in the value of the fraction?

20. Separate 660 into three parts such that the ratio of the first to the second is 5 : 6 and of the second to the third is 9 : 11.

146. Proportion. A *proportion* is a statement of equality between two ratios. Four numbers, a, b, c, and d, are in proportion if the ratio of the first pair equals the ratio of the second pair.

This proportion is written

$$a : b = c : d \text{ or } \frac{a}{b} = \frac{c}{d}.$$

Though both forms are equations, the second is the more familiar one and for this reason is preferable.

NOTE. By the early mathematicians ratios were not treated as if they were numbers, and the equality of two ratios which we know as a proportion was not denoted by the same symbol as other kinds of equality. The usual sign of equality for ratios was ::, a notation which was introduced by the Englishman Oughtred in 1631 and was brought into common use by John Wallis about 1686. The sign = was used in this connection by Leibnitz (1646–1716) in Germany, and by the Continental writers generally, while the English clung to Oughtred's notation.

In the proportion $\frac{a}{b} = \frac{c}{d}$ the first and fourth terms (a, d) are called the *extremes*, and the second and third terms (b, c) are called the *means*.

147. Mean proportional. A *mean proportional* between two numbers a and b is the number m if $\frac{a}{m} = \frac{m}{b}$. It follows that $m^2 = ab$, or $m = \pm\sqrt{ab}$.

If $\frac{4}{m} = \frac{m}{9}$, $m^2 = 36$ and $m = \pm 6$.

148. Third proportional. A *third proportional* to two numbers a and b is the number t if $\frac{a}{b} = \frac{b}{t}$.

A third proportional to 3 and 5 is t if $\frac{3}{5} = \frac{5}{t}$. Whence $t = \frac{25}{3}$.

Gottfried Wilhelm Leibnitz

149. Fourth proportional. A *fourth proportional* to three numbers a, b, and c is the number f if $\frac{a}{b} = \frac{c}{f}$.

A fourth proportional to 6, 8, and 12 is f, if $\frac{6}{8} = \frac{12}{f}$. Whence $f = 16$.

EXERCISES

1. Find a mean proportional between 16 and 25.

2. Find a third proportional to 16 and 25.

3. Find a fourth proportional to 8, 20, and 32.

4. Find the fourth proportional to a^3, ab, and $5\,a^2b$.

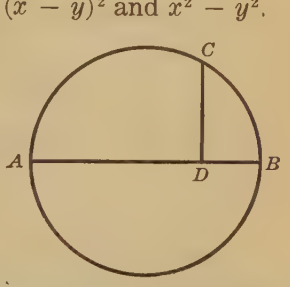
5. Find a mean proportional between $12\,mx^4$ and $3\,m^3$.

6. Find a fourth proportional to 1, $a + b$, and $a^2 - ab + b^2$.

7. Find a mean proportional to $(x^2 + ax + a^2)^2$ and $(x^2 - ax + a^2)^2$.

8. Find a third proportional to $(x - y)^2$ and $x^2 - y^2$.

9. In the adjacent figure CD is perpendicular to the diameter AB. It is a theorem of geometry that CD is a mean proportional between AD and DB. If the radius is 26 and BD is 8, find CD.



10. If four quantities are in proportion, and the third is a mean proportional between the first and second, prove that the second is a mean proportional between the third and fourth.

150. Test of a proportion. Since a proportion is an equality between two ratios (fractions), it is therefore an equation.

Hence any operation which may be performed on an equation may be performed on a proportion. (See Axioms, page 21.)

Thus, in the proportion $\frac{a}{b} = \frac{c}{d}$ both members may be multiplied by bd, giving $ad = bc$. Here the first member is the product of the extremes of the proportion, and the second member is the product of the means.

Therefore in any proportion the product of the extremes equals the product of the means.

151. Proportions from equal products. The numbers which occur in a pair of equal products may be used in various ways as the terms of a proportion.

Thus, if $ad = bc$,

we may write either $\frac{a}{b} = \frac{c}{d}$, or $\frac{a}{c} = \frac{b}{d}$.

Proof. If $a \cdot d = b \cdot c$ is divided by bd, we obtain

$$\frac{ad}{bd} = \frac{bc}{bd}, \text{ or } \frac{a}{b} = \frac{c}{d}. \tag{1}$$

If $a \cdot d = b \cdot c$ is divided by cd, we obtain

$$\frac{a}{c} = \frac{b}{d}. \tag{2}$$

If the means in (1) are interchanged, (2) is obtained. This process of obtaining (2) from (1) is called *alternation.*

If $a \cdot d = b \cdot c$ is divided by ac, we obtain

$$\frac{d}{c} = \frac{b}{a}. \tag{3}$$

If the fractions in (1) are inverted, (3) is obtained. This process of obtaining (3) from (1) is called *inversion.*

EXERCISES

1. Write by alternation (*a*) $\frac{p}{q} = \frac{r}{s}$; (*b*) $\frac{7}{12} = \frac{10\frac{1}{2}}{18}$.

2. Write by inversion (*a*) $\frac{p}{q} = \frac{r}{s}$; (*b*) $\frac{13}{12} = \frac{17\frac{1}{3}}{16}$.

3. Form proportions from the following:

(*a*) $3 \cdot 8 = 2 \cdot 12$.
(*b*) $8 \cdot 7 = 14 \cdot 4$.
(*c*) $uv = xy$.
(*d*) $2x = 3m + 6$.
(*e*) $xy = a$.
(*f*) $a^2 - b^2 = x^2 - y^2$.

4. Find a mean proportional between:

(*a*) 1.44 and .0256; (*b*) $\frac{1}{4}$ and 36;
(*c*) $a^2 - 2ab + b^2$ and m^2.

5. Find a fourth proportional to:

(*a*) 8, 6, and 30; (*b*) 100, 64, and 27;
(*c*) x^2, x^3, and mx.

6. Find a third proportional to:

(*a*) 7 and 40; (*b*) ax and x^3.

Solve for x:

7. $\frac{8}{10} = \frac{x}{5}$. 8. $\frac{12}{13} = \frac{5}{x}$. 9. $x : 8 = 4 : 15$.

10. $7 : 34 = 9 : x$. 11. $\frac{a+b}{a-b} = \frac{c}{x}$.

If $\frac{a}{b} = \frac{c}{d}$, prove the following and state the corresponding theorems in words:

12. $\frac{a}{c} = \frac{b}{d}$. 13. $\frac{b}{a} = \frac{d}{c}$.

14. $\frac{a^n}{b^n} = \frac{c^n}{d^n}$.

15. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}} = \frac{\sqrt[n]{c}}{\sqrt[n]{d}}$.

16. $\frac{a+b}{b} = \frac{c+d}{d}$.

HINT. Add 1 to each member of $\frac{a}{b} = \frac{c}{d}$.

17. $\frac{a+b}{a} = \frac{c+d}{c}$.

HINT. Write $\frac{a}{b} = \frac{c}{d}$ by inversion and add 1 to each member of the result.

18. $\frac{a-b}{b} = \frac{c-d}{d}$.

19. $\frac{a-b}{a} = \frac{c-d}{c}$.

20. $\frac{a+b}{a-b} = \frac{c+d}{c-d}$.

The proportions given in Exercises 16, 18, and 20 are said to be derived from $a : b = c : d$ by *addition*, *subtraction*, and *addition and subtraction*, respectively.

If $\frac{a}{b} = \frac{c}{d}$, show that the following equalities are true:

21. $\frac{5a}{b} = \frac{5c}{d}$.

22. $\frac{2a}{7b} = \frac{2c}{7d}$.

23. $\frac{a^2}{b^2} = \frac{ac}{bd}$.

24. $\frac{5a+b}{5a-b} = \frac{5c+d}{5c-d}$.

25. $\frac{a^2-b^2}{b^2} = \frac{c^2-d^2}{d^2}$.

26. $\frac{a^2-7b^2}{b^2} = \frac{c^2-7d^2}{d^2}$.

27. $\frac{a^2-b^2}{2ab} = \frac{c^2-d^2}{2cd}$.

28. $\frac{7a^2-3b^2}{5ab} = \frac{7c^2-3d^2}{5cd}$.

29. $\frac{a^2+ab+b^2}{c^2+cd+d^2} = \frac{a^2-ab+b^2}{c^2-cd+d^2}$.

30. In the proof which follows give the reason for each step and state the result verbally.

If $\frac{a}{b} = \frac{c}{d} = \frac{e}{f}$, then $\frac{a+c+e}{b+d+f} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}$.

Proof. Setting each of the given ratios above equal to r,

$$\frac{a}{b} = r, \frac{c}{d} = r, \text{ and } \frac{e}{f} = r. \tag{1}$$

Then from (1), $a = br, c = dr, e = fr.$ (2)

Adding in (2), $a + c + e = br + dr + fr.$ (3)

Factoring in (3), $a + c + e = (b + d + f)r.$ (4)

Therefore $$\frac{a + c + e}{b + d + f} = r. \tag{5}$$

Hence by (1) and (5),

$$\frac{a + c + e}{b + d + f} = \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

31. If $\frac{a}{b} = \frac{c}{d} = \frac{u}{v} = \frac{x}{y}$, prove $\frac{a + c + u + x}{b + d + v + y} = \frac{u}{v}$.

32. If $\frac{2}{3} = \frac{10}{15} = \frac{12}{18}$, show that $\frac{2 + 10 + 12}{3 + 15 + 18} = \frac{2}{3}$.

33. If $\frac{a}{b} = \frac{x}{y} = \frac{\sqrt{5}}{\sqrt{7}}$, prove that $\frac{a + x + \sqrt{5}}{b + y + \sqrt{7}} = \frac{a}{b}$.

Solve for x:

34. $(x + 3) : (x - 5) = 12 : 23.$

35. $(3x + 1) : (4x - 3) = 4x : (2x - 1).$

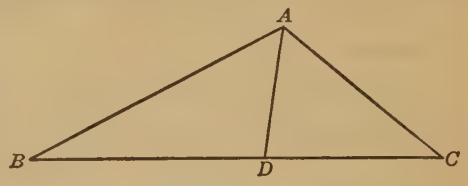
36. $(a + b) : (a - b) = (c + x) : (c - x).$

37. $(a + c + e) : (b + d + x) = a : b.$

PROBLEMS

1. A lamp post 9 feet high casts a shadow 7 feet long. How high is a flag pole which at the same time casts a shadow 42 feet long?

2. The bisector of an angle of a triangle divides the opposite side into segments proportional to the adjacent sides. In triangle ABC if $AB = 18$, $AC = 12$, and $BC = 25$, find the segments of side BC made by the bisector of angle A.



3. The areas of two similar polygons are to each other as the squares of any two corresponding sides. If the corresponding sides of two similar polygons are 30 inches and 42 inches respectively and the area of the second is 1296 square inches, find the area of the first.

4. Two corresponding sides of two similar trapezoids are 18 feet and 28 feet respectively. The area of the first is 196 square feet. Find the area of the second.

5. The area of the surface of a sphere is $4\pi r^2$. If S represents the surface, R the radius, and D the diameter, show that for any two spheres

$$\frac{S_1}{S_2} = \frac{R_1^2}{R_2^2} = \frac{D_1^2}{D_2^2}.$$

6. Compare the areas of two spheres if their radii have the ratio 3 : 5.

7. The diameter of the moon is 2160 miles and of the earth 7920 miles. Find the ratio of their surfaces.

8. The volume of a sphere is $\frac{4\pi R^3}{3}$. If V represents the volume, show that for any two spheres

$$\frac{V_1}{V_2} = \frac{R_1^3}{R_2^3} = \frac{D_1^3}{D_2^3}.$$

9. Compare the volumes of two spheres whose radii are in the ratio 5 : 7.

10. Combining the results of Problems 5 and 8 show that

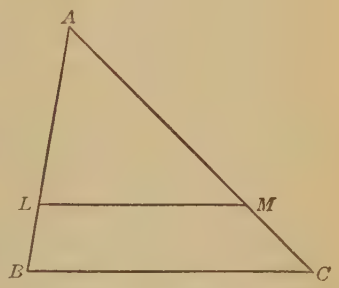
$$\frac{V_1}{V_2} = \frac{S_1^{\frac{3}{2}}}{S_2^{\frac{3}{2}}}.$$

11. The radius of the sun is to the radius of the earth as 109 : 1. Find the ratio of their volumes and the ratio of their surfaces.

12. If ABC is any triangle and LM is a line parallel to BC, meeting AB at L and AC at M, it is proved in geometry that

$$\frac{\text{area } ABC}{\text{area } ALM} = \frac{\overline{AB}^2}{\overline{AL}^2} = \frac{\overline{AC}^2}{\overline{AM}^2} = \frac{\overline{BC}^2}{\overline{LM}^2}.$$

If in the adjacent figure the area of triangle ABC is 196 square inches and AL is 15 inches and AB is 21 inches, find the area of the triangle ALM. What is the area of the trapezoid $BCML$?



13. If in the figure of Exercise 12, AL is 18 and AB is 28 and ALM equals 300, find the area of ABC.

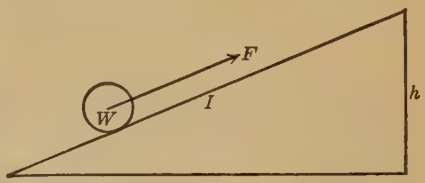
14. If in the figure of Exercise 12, AL is 12 feet and AB is 20 feet, and area ABC equals 400 square feet, find the area of the trapezoid $BCML$.

15. If in the figure of Exercise 12, AL is 60 feet and area ALM : area $BCML = 4 : 5$, find AB.

16. If in the figure of Exercise 12, ABC equals twice $BCML$ and $AB = 100$, find AL to two decimals.

17. Compare the radii of two spheres whose volumes are in the ratio of 64 : 343.

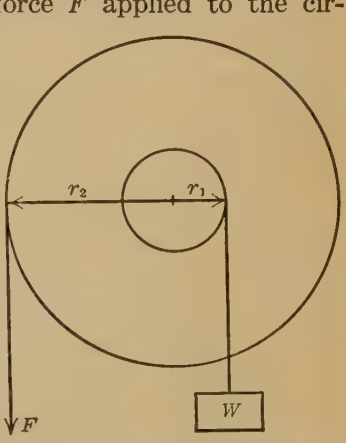
18. It is shown in physics that, ignoring friction, the force F required to hold a weight W on the inclined plane of length I and height h is determined by the proportion $F : W = h : I$. If h is 3 feet and I is 15 feet, what force will be required to hold a weight of 500 pounds?



19. If W is 240 pounds and the ball-bearing carriage carrying it weighs 30 pounds, what force will hold W if h is 5 and $I = 13$?

20. The adjacent figure shows a wheel and axle. It is proved in physics that the force F applied to the circumference of the wheel of radius r_2 will just balance a weight W hung by a rope around the cylinder of radius r_1, provided $F : W = r_1 : r_2$.

If the radius of the wheel is 10 inches and that of the axle is 3 inches, what force will be required to raise a weight of 200 pounds?

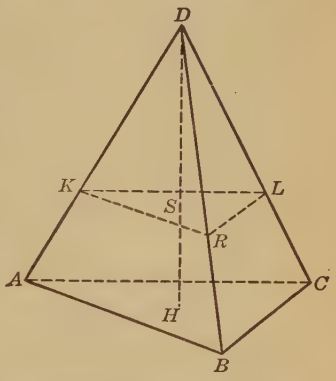


21. The crank of a windlass is 24 inches long and the cylinder on which the rope carrying the bucket winds up is 8 inches in diameter. Find the force on the crank required to lift a bucket of rock weighing 280 pounds.

22. If a plane be passed parallel to the base of a pyramid as in the accompanying figure, cutting it in *KRL*, then

$$\frac{\text{pyramid } DABC}{\text{pyramid } DKRL} = \frac{\overline{DH}^3}{\overline{DS}^3} = \frac{\overline{DA}^3}{\overline{DK}^3}, \text{ etc.}$$

If in the adjacent figure the pyramids are 64 and 125 cubic inches respectively and the altitude *DH* is 50 inches, find *DS*.



23. If the frustum *ABCKRL* is $\frac{26}{27}$ of the whole pyramid and $DS = 24$, find *DH*.

24. If *KRL* divides the pyramid into two equal parts and $DH = 10$, find *DS* and *SH* to two decimals.

152. Variation. The word *quantity* denotes anything which is *measurable*, such as distance, rate, time, and area.

Many operations and problems in mathematics deal with numerical measures of quantities, some of which are fixed and others constantly changing. Such problems as deal with the relation of the numeric measures of at least two changing quantities are called problems in variation.

The theory of variation is really involved in proportion. This fact will become evident after a study of the illustrations of the different types of variation here given.

The equation $x = 5y$ may refer to no physical quantities whatever, yet it is possible to imagine y as taking on in succession every possible numeric value, and the value of x as changing with every change of y, and consequently

always being five times as great as the corresponding value of y. In this sense, which is strictly mathematical, x and y are variables.

153. Direct variation. One hundred feet of iron wire of a certain size weighs 8 pounds. Obviously 200 feet of the same wire would weigh 16 pounds, a piece 300 feet long would weigh 24 pounds, and so on.

Here we have two variables, W(weight) and L(length), so related that the value of W depends on the value of L, in such a way that W increases proportionately as L increases. That is, W is directly proportional to L. Hence if W_1 and W_2 are any two weights corresponding to the lengths L_1 and L_2 respectively

$$W_1 : W_2 = L_1 : L_2. \tag{1}$$

The fact expressed by (1) can be stated in the form of a variation, thus: W varies as L.

In general, if x varies as y, and x and y denote *any* two corresponding values of the variables, and x_1 and y_1 a *particular* pair of corresponding values of these variables,

then $$\frac{x}{x_1} = \frac{y}{y_1}. \tag{2}$$

From (2), $$x = \left(\frac{x_1}{y_1}\right)y. \tag{3}$$

But $\frac{x_1}{y_1}$ is a constant, being the quotient of two definite numbers, while x and y denote any corresponding values of the variables.

Call this constant K, and (3) may be written

$$x = Ky. \tag{4}$$

That is, *if one variable varies as a second, the first always equals the second multiplied by some constant.*

Thus for the iron wire just mentioned, $W = \frac{8}{100} L$, or $\frac{2}{25} L$. Here, though W varies as L varies, W is always equal to L multiplied by the constant $\frac{2}{25}$.

EXERCISES

1. If x varies as y and $x = 5$ when $y = 9$, find y when x is 16.

Solution. The variation is direct.

Therefore, $$\frac{5}{16} = \frac{9}{y}.$$

Solving, $$y = 28\tfrac{4}{5}.$$

2. If x varies as y and $x = 12$ when $y = 35$, find y when $x = 27$.

3. If x varies as y and $x = n$ when $y = v$, find x when y is m.

4. If x varies as y and $x = 6$ when $y = 15$, find k in $x = ky$.

5. A certain quantity of gas is inclosed in a gas holder. If the volume of the gas varies as the temperature and the volume is 500 cubic feet when the temperature (absolute) is 300°, what is the volume when the temperature rises 30°? when it falls 100°?

6. If the volume of gas in a gas holder is 1,000,000 cubic feet at 300°, find the change in temperature if it causes an increase in volume of 100,000 cubic feet.

7. One quantity may vary directly as the square root of another. Express this relation in symbols: (*a*) as a variation, (*b*) as a proportion.

8. One quantity may vary as the cube of another. Express this relation in symbols: (*a*) as a variation, (*b*) as a proportion.

154. Inverse variation. If two variables are so related that one increases as the other decreases and vice versa, either variable is said to vary inversely as the other. For example, if the area of a triangle, A, is constant while the base, b, and altitude, a, vary, the base varies inversely as the altitude. This is apparent from the equation $A = \frac{1}{2}ab$.

Similarly if d is the distance covered by a motor bus each trip, and r its speed, and t the time, $d = rt$. Here for a fixed route of length d, t varies inversely as r.

Therefore $$t_1 : t_2 = \frac{1}{r_1} : \frac{1}{r_2}. \tag{5}$$

In the form of a variation (5) becomes t varies as $\frac{1}{r}$.

In general, x varies *inversely as* y when x varies as the *reciprocal of* y; that is,

$$x \text{ varies as } \frac{1}{y}. \tag{6}$$

And if x and y denote any two corresponding values of the variable, and x_1 and y_1 a particular pair of corresponding values,

$$x : x_1 = \frac{1}{y} : \frac{1}{y_1}. \tag{7}$$

Whence $$\frac{x}{y_1} = \frac{x_1}{y}, \text{ or } xy = x_1y_1. \tag{8}$$

But x_1y_1 is a constant, being the product of two definite numbers. Call this constant K.

Then (8) becomes $xy = K$.

That is, if one variable varies inversely as another, the product of the two is a constant.

Many phenomena in nature are governed by a law which involves inverse variation, and the square or the cube of one of the variables concerned. Thus the force of gravitation between two bodies varies inversely as the square of the distance between them. And the force with which a magnet attracts a piece of soft iron varies inversely as the square of the distance between them. Also the intensity of light or sound varies inversely as the square of the distance of the observer from the source of the light or the sound. The tide-producing force of the moon on the ocean varies inversely as the cube of the moon's distance from the earth.

EXERCISES

1. If x varies inversely as t and $x = 10$ when t is 2, find x when t is 60.

Solution. The variation is inverse.

Therefore, $10 : x = \frac{1}{2} : \frac{1}{60}$.

Solving, $x = \frac{1}{3}$.

2. If x varies as $\frac{1}{y}$ and $x = 3$ when $y = 8$, find x when y is 4.

3. If y varies as $\frac{1}{t}$ and $y = u$ when $t = x$, find y when $t = m$.

4. If s varies as $\frac{1}{d}$ and $s = 4$ when $d = 6$, find d when $s = 20$.

5. If the pressure on a certain quantity of gas in a gas holder is changed without changing the temperature, then Boyle's law holds. This is sometimes stated $p_1v_1 = p_2v_2$, where p represents pressure and v the volume. State this equation as a proportion. State it as a variation.

6. Six hundred cubic feet of gas in a gas holder is under a pressure of 24 pounds to the square inch. Use Boyle's law and determine the volume if without changing the temperature the pressure is changed to (*a*) 20 pounds; (*b*) 30 pounds.

7. The pressure on 3600 cubic feet of air at 60 pounds per square inch is changed so that the volume becomes 2400 cubic feet. Find the pressure.

8. One quantity may vary inversely as the square root of another. Express such a relation in symbols (*a*) as a variation, (*b*) as a proportion.

9. One quantity may vary inversely as the square or the cube of another. Express each of these relations in symbols (*a*) as a proportion, (*b*) as a variation.

155. Joint variation. One quantity is said to vary jointly as two others when it varies directly as their product. For example, the wages of a workman varies jointly with the amount he receives per hour and the number of hours he works. If W represents his total wages, p the amount received per hour, and n the number of hours he works, $W = np$. Similarly the formula for the area of a rectangle $A = ab$ is an example of joint variation. One variable often varies jointly as the product of more than two variables.

Thus the pressure of a horizontal wind on a roof varies jointly as the square of the wind's velocity, the area of the roof, and the cosine of the angle which the roof makes with the vertical.

In general, any variable x varies jointly as two others, y and z, if

$$x \text{ varies as } yz; \tag{1}$$

that is, if x varies as the product of the two.

If x varies jointly as y and z, and if x, y, and z denote any corresponding values of the variables, while x_1, y_1, and z_1 denote a particular set of such values, then

$$\frac{x}{x_1} = \frac{yz}{y_1z_1}. \tag{2}$$

From (2), $$x = \left(\frac{x_1}{y_1z_1}\right)yz. \tag{3}$$

But in (3) the fraction $\frac{x_1}{y_1z_1}$ is a constant, since x_1, y_1, and z_1 are particular values of the variables x, y, and z. Calling this constant K, we may write x varies as yz as the equation

$$x = Kyz.$$

One variable may vary directly as another variable (or several variables) and inversely as still another (or several others). Also one variable may vary as the square, or the cube, or the square root, or the reciprocal, or as any algebraic expression whatever involving the other variable (or variables).

EXERCISES

1. If x varies jointly as y and z, and $x = 30$ when $y = 8$ and $z = 10$, find x when $y = 18$ and $z = 4$.

Solution. The variation is joint.

Therefore, $$\frac{30}{x} = \frac{8 \cdot 10}{18 \cdot 4}.$$

Solving, $$x = 27.$$

2. If x varies as uv, and $x = 300$ when $u = 35$ and $v = 24$, find x when $u = 100$ and $v = 60$.

3. If A varies as hb, and $A = 84$ when $h = 16$ and $b = 50$, find A when $h = 30$ and $b = 40$.

PROBLEMS

1. The intensity (brightness) of light varies inversely as the square of the distance from the source of light. A book is held 2 feet from a lamp and later 5 feet distant. At which distance does it appear brighter? How many times as bright?

2. The intensity of sound varies inversely as the square of the distance from the source of the sound. Compare the intensity of the sound of a bell for a listener who is at first 300 feet away and later is 2 miles distant.

3. The safe load, W, for a beam varies jointly as the thickness t, the depth d, and inversely as the length l between the supports. Express this relation between W, t, d, and l by a proportion.

4. A safe load of 2000 pounds is carried by a beam 16 feet long, 4 inches thick, and 18 inches deep. What is the safe load for a beam of the same material half as long, half as thick, and half as deep?

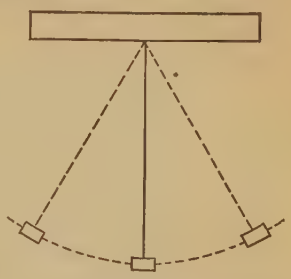
5. If x varies as y and inversely as z, represent this relation in symbols by a proportion.

6. If x varies as y^2 and inversely as the square root of z, represent this relation in symbols by a proportion and by a variation.

7. The volume of a given weight of gas varies inversely as the pressure and directly as the temperature. Express this relation between v, p, and t by a proportion and by a variation.

8. If the volume is 600 when $p = 32$ and $t = 320$, find the volume when $p = 48$ and $t = 360$.

9. The time of vibration of a pendulum varies directly as the square root of its length divided by the force of gravitation. Express the relation between t, l, and g by a proportion.



10. If a pendulum 100 centimeters long vibrates once in 1 second, find the period of a pendulum 121 centimeters long.

11. If a pendulum 100 centimeters long vibrates once in one second at sea level where the force of gravitation is 982, find the time of one vibration of a pendulum 144 centimeters long on top of a mountain where the force of gravitation is 970.

12. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. Express this relation as a variation and also as a proportion.

13. An object weighing 160 pounds at the earth's surface would weigh how much 800 miles above the surface? 2000 miles above the earth's surface? (Radius of earth equals 4000 miles.)

14. A meteorite weighs one ton. How far from the earth's surface was it when its weight was half as much? one tenth as much?

15. The weight of any object below the surface of the earth varies directly as its distance from the center. An object weighs 100 pounds at the earth's surface. Find its weight (*a*) 800 miles below the surface, (*b*) 2000 miles below the surface, (*c*) at the center. (Radius of earth = 4000 miles.)

16. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. Express this relation as a variation and as a proportion.

17. If the pressure of the wind blowing 20 miles per hour is .8 of a pound on one square foot, find the pressure on one wall of a building 100 feet long and 120 feet high.

18. Newton's law of gravitation states that the force of attraction between two bodies (spheres) varies as the product of their masses and inversely as the square of the distance between them. Express this relation as a variation and as a proportion.

19. The mass of the sun is 330,000 times the mass of the earth and the distance between them (that is, the distance between their centers) is 93,000,000 miles. The force of attraction between the sun and the earth is 36×10^{17} tons. If the earth's mass is 81 times the mass of the moon, find the attraction between the earth and the moon if the distance between them is 240,000 miles.

CHAPTER XXI

DETERMINANTS

156. Determinants of the second order. The arrangement of numbers $\begin{vmatrix} 5 & 2 \\ 4 & 3 \end{vmatrix}$ has been given the meaning $5 \cdot 3 - 4 \cdot 2$, which equals 7.

Such an arrangement is called a *determinant.* Since it has two rows and two columns, it is said to be of the *second order.*

A determinant of the second order really represents a number, and this number is easily found since

$$\begin{vmatrix} a & c \\ b & d \end{vmatrix} = ad - bc.$$

Accordingly $\begin{vmatrix} 7 & 3 \\ 6 & 4 \end{vmatrix} = 7 \cdot 4 - 3 \cdot 6 = 28 - 18 = 10.$

Similarly $\begin{vmatrix} 5 & -6 \\ 2 & 3 \end{vmatrix} = 5 \cdot 3 - 2(-6) = 15 + 12 = 27.$

The preceding operations can be reversed and the difference or sum of two products written as a determinant of the second order.

Thus $pq - uv$ can be written $\begin{vmatrix} p & v \\ u & q \end{vmatrix}$ or $\begin{vmatrix} p & u \\ v & q \end{vmatrix}$ or $\begin{vmatrix} q & v \\ u & p \end{vmatrix}$ or $\begin{vmatrix} q & u \\ v & p \end{vmatrix}$ since each equals $pq - uv$.

Similarly $mn - r = m \cdot n - 1 \cdot r = \begin{vmatrix} m & r \\ 1 & n \end{vmatrix}$.

Also $3 \cdot 7 - 2 \cdot 4 = \begin{vmatrix} 3 & 4 \\ 2 & 7 \end{vmatrix}$.

EXERCISES

Find the value of the determinants:

1. $\begin{vmatrix} 4 & 1 \\ 2 & 3 \end{vmatrix}$.
2. $\begin{vmatrix} 5 & -2 \\ 3 & 4 \end{vmatrix}$.
3. $\begin{vmatrix} 11 & 9 \\ 4 & 7 \end{vmatrix}$.
4. $\begin{vmatrix} 8 & -3 \\ 2 & 5 \end{vmatrix}$.
5. $\begin{vmatrix} 6 & 5 \\ 7 & 3 \end{vmatrix}$.
6. $\begin{vmatrix} 18 & 4 \\ 12 & 3 \end{vmatrix}$.
7. $3\begin{vmatrix} 15 & 4 \\ 18 & -3 \end{vmatrix}$.
8. $4a\begin{vmatrix} -9 & 68 \\ 14 & 16 \end{vmatrix}$.
9. $m\begin{vmatrix} 15 & 11 \\ -6 & 8 \end{vmatrix}$.
10. $\frac{7}{3m}\begin{vmatrix} m^3 & 0 \\ m^2 & 2m \end{vmatrix}$.
11. $ax\begin{vmatrix} (a-x) & a \\ (a+x) & x \end{vmatrix}$.
12. $\frac{-4}{3}\begin{vmatrix} (a+b) & (a-b) \\ (a-b) & (a+b) \end{vmatrix}$.

Write as a determinant:

13. $mx - ny$.
14. $an - bc$.
15. $az + xy$.
16. $ab - m$.
17. $cd - ab$.
18. $x - y$.

Write as the quotient of two determinants:

19. $\frac{ab - cd}{pq - ml}$.
20. $\frac{am - 15}{2x - 2y}$.
21. $\frac{\frac{x}{2} - 6}{3a - 1}$.
22. $\frac{5a - xy}{4x - 3a}$.
23. $\frac{x^2 - y^2}{x^2 + y^2}$.
24. $\frac{ax + ay}{bc - ad}$.

157. Solution by determinants. The foregoing principles are easily applied to the solution of any linear system in two unknowns by means of determinants of the second order.

The ordinary solution of the system $ax + by = c$ (1)

$$dx + ey = f \quad (2)$$

$$\text{gives } x = \frac{ce - bf}{ae - bd} \text{ and } y = \frac{af - cd}{ae - bd}.$$

$$\text{Using determinants } x = \frac{ce - bf}{ae - bd} = \frac{\begin{vmatrix} c & b \\ f & e \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}} \quad (3)$$

$$\text{and } y = \frac{af - cd}{ae - bd} = \frac{\begin{vmatrix} a & c \\ d & f \end{vmatrix}}{\begin{vmatrix} a & b \\ d & e \end{vmatrix}}. \quad (4)$$

If the denominators in (3) and (4) are zero and the numerators are not zero, then no set of roots exists for (1) and (2).

The determinant expressions for x and y in (3) and (4) can be used as general formulas to solve any pair of linear equations in two unknowns. The determinant forms can be easily remembered and written down at once if we observe the following points:

I. *The determinants in the denominators are identical, and each is formed by the coefficients of* x *and* y *as they stand in the original equations* (1) *and* (2).

II. *The determinant in the numerator of the value of* x *is formed from the denominator by replacing the coefficients of* $\begin{matrix} a \\ d \end{matrix}$, *by the constant terms* $\begin{matrix} c \\ f \end{matrix}$.

III. *The determinant in the numerator of the value of* y *is formed from the denominator by replacing the coefficients of* $y, \begin{matrix} b \\ e \end{matrix}$, *by the constant terms* $\begin{matrix} c \\ f \end{matrix}$.

EXAMPLE

Solve by determinants: $\begin{cases} 3y + 13x = 14, \\ 7x - 2y = 22. \end{cases}$

Solution. Writing the equations in standard form, we have

$$\begin{cases} 13x + 3y = 14, \\ 7x - 2y = 22. \end{cases}$$

Then $$x = \frac{\begin{vmatrix} 14 & 3 \\ 22 & -2 \end{vmatrix}}{\begin{vmatrix} 13 & 3 \\ 7 & -2 \end{vmatrix}} = \frac{-28 - 66}{-26 - 21} = \frac{-94}{-47} = 2.$$

In solving for y the denominator is the same as before.

Hence $$y = \frac{\begin{vmatrix} 13 & 14 \\ 7 & 22 \end{vmatrix}}{-47} = \frac{286 - 98}{-47} = \frac{188}{-47} = -4.$$

Check as usual.

EXERCISES

Solve by determinants and check results:

1. $5x + 2y = 16,$
   $3x + 4y = 18.$

2. $4x + y = 2,$
   $x - 2y = 5.$

3. $7h - k = 2,$
   $6h - k = 3.$

4. $3x = 4y + 14,$
   $3y = 4x - 14.$

5. $6s + 8t = 26,$
$5s - 3t = 70.$

6. $4x + \frac{2z}{3} = \frac{26}{3},$
$3z + \frac{7x}{2} + 4 = 0.$

7. $\frac{5x}{6} + \frac{y}{4} = 7,$
$\frac{2x}{3} - \frac{y}{8} = 3.$

8. $4x + 2y = m,$
$2x + 4y = n.$

9. $my + 3z = 3 + 6m,$
$m^2y + z = 5m.$

10. $au + bv = a + b,$
$au - bv = a - b.$

11. $\frac{u}{m} + \frac{v}{n} = \frac{m+n}{mn},$
$u - v = \frac{m^2 - n^2}{mn}.$

12. $ay + (b + c)z = 1,$
$az = 1 - (b + c)y.$

158. Determinants of the third order. By definition any arrangement of numbers like:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} =$$

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1. \qquad (1)$$

Such an arrangement is called a determinant of the *third order* because it has three *rows* and three columns. Each of the nine numbers in the determinant is called an *element.*

Grouping terms in the right member of (1), we get

$$(a_1b_2c_3 - a_1b_3c_2) + (a_2b_3c_1 - a_2b_1c_3) + (a_3b_1c_2 - a_3b_2c_1), \qquad (2)$$

or $$a_1(b_2c_3 - b_3c_2) - a_2(b_1c_3 - b_3c_1) + a_3(b_1c_2 - b_2c_1). \qquad (3)$$

and $$a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2\begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3\begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}. \qquad (4)$$

From (1), (2), (3), and (4)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 \\ b_3 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 \\ b_2 & c_2 \end{vmatrix}. \quad (5)$$

Equation (5) shows that a determinant of the third order can be expressed in terms of three determinants of the second order, the elements of the latter being elements of the third-order determinant.

The second-order determinants of (5) can always be easily written because their method of formation is governed by two principles:

I. *The coefficients of the second-order determinants are the elements of the first column of the third-order determinant, the sign of* a_2 *being minus.*

II. *The elements of the second-order determinants are those* not *in the same row or column with the coefficients* a_1, a_2, a_3. *Thus the elements of* $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$ *are not in the first row or column as is the coefficient* a_1.

Hence $$\begin{vmatrix} 2 & 3 & 4 \\ 5 & 1 & 6 \\ 3 & 7 & 4 \end{vmatrix} = 2 \begin{vmatrix} 1 & 6 \\ 7 & 4 \end{vmatrix} - 5 \begin{vmatrix} 3 & 4 \\ 7 & 4 \end{vmatrix} + 3 \begin{vmatrix} 3 & 4 \\ 1 & 6 \end{vmatrix}$$

$$= -76 + 80 + 42 = 46.$$

EXERCISES

Find the value of:

1. $\begin{vmatrix} 1 & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 5 & 3 \end{vmatrix}$.

2. $\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & -1 \\ 3 & -1 & 1 \end{vmatrix}$.

3. $\begin{vmatrix} 5 & -1 & 1 \\ 6 & -2 & 0 \\ 0 & 2 & 3 \end{vmatrix}$.

4. $12\begin{vmatrix} 3 & -2 & 4 \\ 2 & 3 & -2 \\ 5 & 2 & 3 \end{vmatrix}$.

5. $\begin{vmatrix} 9 & 5 & 4 \\ 5 & 6 & 5 \\ 16 & 4 & 6 \end{vmatrix}$.

6. $\begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 3 \\ 1 & 0 & 8 \end{vmatrix}$.

7. $\begin{vmatrix} 5 & -7 & 4 \\ 6 & 3 & -5 \\ 4 & 6 & 3 \end{vmatrix}$.

8. $\begin{vmatrix} x & y & 1 \\ 3 & 4 & 1 \\ -8 & 6 & 1 \end{vmatrix}$.

9. $\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$.

10. $\begin{vmatrix} a & 3 & 4 \\ b & 4 & 5 \\ c & 5 & 6 \end{vmatrix}$.

11. $bc\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix}$.

12. $\begin{vmatrix} (a+b) & 2 & 1 \\ 2 & (a-b) & 1 \\ 3 & 3 & 1 \end{vmatrix}$.

159. The general linear system in three variables. The method of addition and subtraction applied to the system

$$\begin{cases} a_1x + b_1y + c_1z = d_1, & (1) \\ a_2x + b_2y + c_2z = d_2, & (2) \\ a_3x + b_3y + c_3z = d_3, & (3) \end{cases}$$

gives $$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_3b_2c_1 - d_1b_3c_2 - d_2b_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \quad (4)$$

$$y = \frac{a_1d_2c_3 + a_2d_3c_1 + a_3d_1c_2 - a_3d_2c_1 - a_1d_3c_2 - a_2d_1c_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}, \quad (5)$$

$$z = \frac{a_1b_2d_3 + a_2b_3d_1 + a_3b_1d_2 - a_3b_2d_1 - a_1b_3d_2 - a_2b_1d_3}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_3b_2c_1 - a_1b_3c_2 - a_2b_1c_3}. \quad (6)$$

By grouping and factoring as in section 158 the values (4), (5), and (6) become

(7) (8) (9)

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

The fractions (4), (5), and (6) are general results, and can be used as formulas to solve any three simultaneous equations in three variables, but the equivalent forms (7), (8), and (9) are far more easily remembered. These can be written down at once for any system of three equations in three variables since

I. *The determinants in the denominators are identical and each is formed by the coefficients of* x, y, z, *as they stand in the original equations.*

II. *Each determinant in the numerator is formed from the denominator by putting the column of constant terms (as they stand in the original system) in place of the column of the coefficients of the variable whose value is sought.*

If the denominator of (7), (8), and (9) is zero and the numerators are not zero, then no set of roots exist for (1), (2), and (3).

EXAMPLE

Solve, using determinants:

1.

$$\begin{cases} x + 3y = 2 - 5z, & (1) \\ 2x + z = y + 1, & (2) \\ 3x + 5y + 7z = -10. & (3) \end{cases}$$

Solution. Rewriting in standard form,

$$x + 3y + 5z = 2, \tag{4}$$
$$2x - y + z = 1, \tag{5}$$
$$3x + 5y + 7z = -10. \tag{6}$$

From I and II, page 326,

$$x = \frac{\begin{vmatrix} 2 & 3 & 5 \\ 1 & -1 & 1 \\ -10 & 5 & 7 \end{vmatrix}}{\begin{vmatrix} 1 & 3 & 5 \\ 2 & -1 & 1 \\ 3 & 5 & 7 \end{vmatrix}} = -\frac{100}{20} = -5$$

$$y = \frac{\begin{vmatrix} 1 & 2 & 5 \\ 2 & 1 & 1 \\ 3 & -10 & 7 \end{vmatrix}}{20} = -\frac{120}{20} = -6.$$

The value of z can now be more easily obtained by substituting the values of x and y already found in (1), (2), or (3) than by determinants.

Substituting in (2), $-10 + z = -6 + 1$.

Hence, $z = 5$.

Check as usual.

EXERCISES

Solve for x, y, and z as in the preceding example and check results.

1. $x + y + z = 27$,
   $x - y + z = 9$,
   $x + y - z = 5$.

2. $2x + y - z = 3$,
   $x - 3y + z = 2$,
   $x + y - 5z = -8$.

3. $3x + y = 22$,
   $x + y - 3z = -7$,
   $x + 3z + y = -11$.

HINT. The first equation is $3x + y + 0z = 22$. The coefficient zero must never be omitted in using determinants.

4. $3x + y = 25,$
$x + 2y = 5,$
$2y + z = 2.$

5. $x + y = 2a,$
$x + z = 3a,$
$y + z = 4a.$

6. $.4x + .3y - 8z = 4,$
$.5x + z + .8y = 1.2,$
$2.6z + .3 - x = .5y.$

7. $3x + 2z = 6a - 2y,$
$x - 5y + 6z = 2a - 11b,$
$6x - 8z = 12a + 8b.$

8. $ax + by = 0,$
$cx - by = 2bc,$
$bx + az - cy = b^2.$

9. $ax + by - cz = 2ab,$
$ay - bx + cz = 2bc,$
$ax - by + cz = 2ac.$

160. Determinants of the fourth order. Linear systems in four unknowns lead to determinants of the fourth order, linear systems in five unknowns lead to determinants of the fifth order, and so on. To find the value of a fourth-order determinant we first express it as four determinants of the third order. Each of these can then be evaluated as in section 159.

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_3 & c_3 & d_3 \\ b_4 & c_4 & d_4 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_4 & c_4 & d_4 \end{vmatrix} - a_4 \begin{vmatrix} b_1 & c_1 & d_1 \\ b_2 & c_2 & d_2 \\ b_3 & c_3 & d_3 \end{vmatrix}.$$

Note that the signs of the coefficients are alternately plus and minus. Similarly a fifth-order determinant can be ex-

pressed in terms of five determinants of the fourth order and so on. In general a determinant of order n can be expressed in terms of n determinants of order $n - 1$.

EXERCISES

Find the value of:

1. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & 5 & 1 \end{vmatrix}$.

2. $\begin{vmatrix} 1 & 4 & 3 & 8 \\ 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 0 \\ 0 & 4 & 1 & 1 \end{vmatrix}$.

3. $\begin{vmatrix} 1 & 1 & 1 & 1 \\ 2 & 2 & 3 & 4 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 5 \end{vmatrix}$.

4. $\begin{vmatrix} x & y & z & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 2 & 2 & 1 \\ 0 & 1 & 0 & 1 \end{vmatrix}$.

5. $\begin{vmatrix} x & y & z & 1 \\ 1 & 1 & 1 & 1 \\ 4 & -3 & -1 & 1 \\ -4 & -1 & -1 & 1 \end{vmatrix}$.

6. $\begin{vmatrix} x & y & z & 1 \\ 10 & 10 & 1 & 1 \\ -8 & -8 & -8 & 1 \\ 6 & 0 & 0 & 1 \end{vmatrix}$.

NOTE. Like so many other discoveries, the determinant notation was noticed independently by two men. In a letter to a friend, writing in 1693, Leibnitz outlined the method of solving equations by means of determinants, but, so far as we know, he used the notation in his own work very little, and certainly did not publish it during his lifetime. In fact, the letter in which this reference is found, did not come to light until 1850, and the fact that Leibnitz knew anything about determinants was not generally recognized until after that time.

In 1750 Cramer, a professor in the university at Geneva, rediscovered this method of solving linear systems, and his work had the good fortune to be accepted by scholars, forming the real beginning of the development of the subject. Since that time a great many have written on the subject, and to-day determinants are used in every field of advanced mathematics.

CHAPTER XXII

LIMITS AND INFINITY

161. Definition of a variable. A number which can take on successively different numeric values is called a *variable.* It should be remembered that, although it is common for the successive values of the variable to be related to each other in accordance with some law, these values need not have any definite relation one to another. As stated in section 152 a variable is commonly represented by a letter such as x, y, or z.

Thus in the equation

$$ax^2 + bx + c = 0,$$

the variable is x. In this case the successive values of the variable are related to each other by the law expressed in the algebraic equation.

162. Limits. The numeric value V of the recurring decimal $.9999\cdots$ is a variable depending on the number of 9's added to the series on the right. Every 9 thus added increases V, and the number of 9's which may be so added is unlimited. In spite of this, V always remains less than 1, although approaching more and more closely to that value. Here the number 1 is called the *limit* of the variable V.

163. Definition of a limit. If a variable V takes on successively a series of values which approach closer and closer to a fixed number L in such a manner that the nu-

meric value of $V - L$ becomes and remains less than any finite number however small, then V is said to approach the *limit* L.

This may be written *limit of* $V = L$.

The symbol $\rightarrow$ gives us the equivalent notation $V \rightarrow L$, which is read V *approaches* L *as a limit.*

164. Infinity. If a variable n takes on in succession all the values 1, 2, 3, 4, ···, we can conceive of no final value for n, since the system of natural numbers is unlimited. Here we may say that n *increases without limit* or n becomes *infinite.* This is stated in the following

DEFINITION. *If a variable* n *becomes and remains greater than any positive number* k, *however great, we say* n *increases without limit, or* n *becomes infinite.*

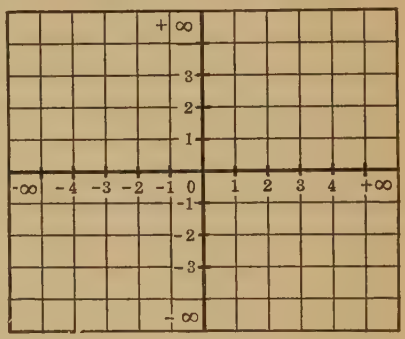
The usual symbol for a variable which has become infinite is the sign ∞, read *infinity.*

Infinity is not a number in the sense in which 3, $\sqrt{7}$, and -9 are numbers. It is greater than any number. For present purposes it must be regarded as a figure of speech rather than as a number that can be added, subtracted, multiplied, or divided, as are ordinary numbers. In fact, the symbol ∞ cannot be operated upon according to the laws governing the fundamental operations as can numbers.

NOTE. Some idea of the reason why we cannot regard ∞ as a number, and operate with it as we do with ordinary numbers, may be seen if we consider all numbers, 1, 2, 3, 4, 5, ···. Evidently this series may be continued as far as we wish, but the number of terms in such a series cannot be expressed by any integer, for it is greater than any number; that is, it is infinite. If we multiply each number in the above series by 2, we have not changed the number of terms in the series, but we have formed the series 2, 4, 6, 8, 10, ···, which includes only even integers, and is therefore included in the original series. We have, then, two infinities which are equal, or at least they are represented by the same symbol, one of which contains the other.

This is contrary to the axiom which we always assume for finite numbers, namely, that the whole is greater than any of its parts. Surely it is not strange that we cannot operate freely with a symbol which violates this fundamental principle.

We may have a negative infinity as well as a positive one. In order to indicate the range of values which x and y may take in graphical work, the axes are often marked as in the adjacent figure.



A *constant* number, however large, is never spoken of as infinite.

If the variable n in $\frac{1}{n}$ takes on in succession the values 1, 2, 3, 4, ···, no final value of $\frac{1}{n}$ can be imagined. But as the variable n increases without limit, $\frac{1}{n}$ becomes very small and approaches nearer and nearer to zero without ever actually becoming zero.

In general, if a in the fraction $\frac{a}{n}$ is any constant not zero, and n a variable increasing without limit, $\frac{a}{n}$ approaches zero as a limit. Unfortunately, in elementary mathematics there is not in general use a symbol for a variable whose limit is zero, though such a symbol would be a great convenience.

The student will frequently meet the statement $\frac{a}{\infty}$ equals zero. This statement is, of course, meaningless until it has been defined, but it may properly be regarded as a way of saying that $\frac{a}{n}$ approaches zero as a limit when n is indefinitely increased.

165. Interpretation of $\frac{a}{0}$. Division by zero is excluded from elementary mathematics for two reasons:

(a) It is never necessary.

(b) It would give rise to endless difficulties and ambiguities.

Results of the form $\frac{a}{0}$, where a is a constant not zero, frequently arise. According to the rules of computation, however, such an expression has no meaning. Though it is true that $\frac{a}{0}$ is not a definite number, results of this form may sometimes admit of interpretation.

As an illustration of this, consider the following

EXAMPLE

Solve by determinants the system $\begin{cases} 3x - 6y = 5, & (1) \\ x - 2y = 1. & (2) \end{cases}$

Solution.
$$x = \frac{\begin{vmatrix} 5 & -6 \\ 1 & -2 \end{vmatrix}}{\begin{vmatrix} 3 & -6 \\ 1 & -2 \end{vmatrix}} = \frac{-10+6}{-6+6} = \frac{-4}{0},$$

and
$$y = \frac{\begin{vmatrix} 3 & 5 \\ 1 & 1 \end{vmatrix}}{0} = \frac{3-5}{0} = \frac{-2}{0}.$$

The graphs of (1) and (2) are parallel lines. For such lines there is no point of intersection and consequently the system has no set of roots. Now as the results for x and y are of the form $\frac{a}{0}$, the attempt at solution by determinants fails. Therefore the interpretation of these results is that no set of roots exists for the system (1), (2).

In general, if for any system of linear equations the results obtained are of the form $\frac{a}{0}$, the system has no set of roots; that is, the system is inconsistent.

If the student meets the statement $\frac{a}{0} = \infty$, he should regard it as a loose use of the statement that $\frac{a}{n}$ becomes infinitely large as n becomes infinitesimally small.

166. Interpretation of $\frac{0}{0}$. The fraction $\frac{x-3}{x^2-9}$ becomes $\frac{0}{0}$ when $x = 3$. For any value of x other than the critical value 3, the fraction equals a definite number. Usually we are concerned with the limit of such expressions as the variable approaches a critical value. The limit for the fraction $\frac{x-3}{x^2-9}$ is easily found. We assign to x successively the values 2.9, 2.99, 2.999, 2.9999, etc. The corresponding values of the fractions are $\frac{10}{59}$, $\frac{100}{599}$, $\frac{1000}{5999}$, $\frac{10000}{59999}$, etc. Obviously these numbers approach the limit $\frac{1}{6}$.

We may arrive at this result more easily as follows: for all values of x except 3 the terms of the fraction $\frac{x-3}{x^2-9}$ may be divided by $(x-3)$, obtaining $\frac{1}{x+3}$. This result is true, however little x may differ from 3. Now if, without giving x the value 3, we make it approach 3 as a limit, $\frac{1}{x+3}$ will approach $\frac{1}{6}$ as a limit, and this is the limit of the original fraction $\frac{x-3}{x^2-9}$ as well.

By either of the preceding methods it can be shown that $\frac{x^2-16}{x-4}$, which becomes $\frac{0}{0}$ for $x = 4$, has 8 as its limit. These two fractions are simple illustrations of the important fact that the symbol $\frac{0}{0}$ is not a definite number. The truth of this can be seen more clearly from a study of the following graph.

EXAMPLE

Graph: $\dfrac{x^2-16}{x-4}$.

Let $$y=\frac{x^2-16}{x-4}. \tag{1}$$

Then $$y(x-4)=(x+4)(x-4). \tag{2}$$

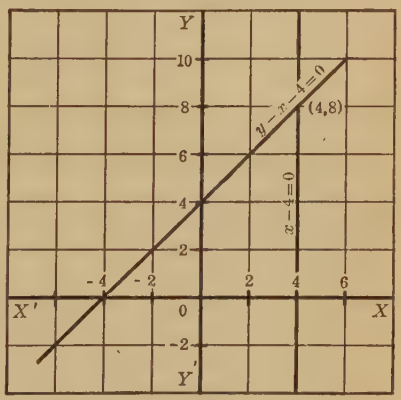
$$y(x-4)-(x+4)(x-4)=0. \tag{3}$$

$$(y-x-4)(x-4)=0. \tag{4}$$

Therefore $$x-4=0, \tag{5}$$

and $$y-x-4=0. \tag{6}$$

The graphs of (5) and (6) are given in the following figure. To understand what follows it must be borne clearly in mind that y and the fraction $\dfrac{x^2-16}{x-4}$ are *identical*, and that the graphs (5) and (6) are the complete graph of the equation (1).



For every value of x except 4 there is always one value of y, which is the y-distance of a point on line (6). For $x=4$, however, the value of y is *any* point on line (5). Hence the fraction $\dfrac{x^2-16}{x-4}$ is indeterminate for $x=4$.

It is interesting to note here that the limiting value of the fraction $\dfrac{x^2-16}{x-4}$ is seen from the graph to be 8, which is the y-value of the point of intersection of lines (5) and (6).

NOTE. The study of the limiting value of the ratio of two functions which for certain values of the variable takes on the indeterminate form $0/0$ was undertaken by the Frenchman L'Hospital, in 1696, and was carried further by John Bernouilli a few years later. A complete comprehension of the difficulties which surround this subject has been very slowly gained by mathematical writers, and even today it is possible to find books in which grave errors are made regarding the meaning of these expressions.

The questions involved are closely related to those regarding the nature of the infinite in mathematics. The penetration of this mystery is one of the great achievements of the latter half of the nineteenth century, and today well-informed mathematicians have as clear and satisfactory ideas about infinite numbers as they do about ordinary integers.

As a final illustration that $\frac{0}{0}$ may have any value, consider the following:

EXAMPLE

Solve by determinants the following system:

$$x+y+z=0, \quad (1)$$
$$x+2y+3z=0, \quad (2)$$
$$4x+y-2z=0. \quad (3)$$

Solution. Solving by determinants, x, y, and z all equal $\frac{0}{0}$. But the system is satisfied by many values of x, y, and z. For example, consider the following sets of values for the three variables:

x	0	-1	1	-2	2
y	0	2	-2	4	-4
z	0	-1	1	-2	2

, etc.

In general, if the solution of a system of linear equations in two or more unknowns gives $\frac{0}{0}$ as values of the un-

knowns, the system has an infinite number of sets of roots; that is, the system is indeterminate.

The symbol $\frac{0}{0}$ then is a symbol of indetermination.

EXERCISES

Solve by determinants and interpret the results:

1. $2x - y = 1,$
   $4x - 2y = 2.$

2. $x - y = 1,$
   $y - x = 0.$

3. $x + y + z = 3,$
   $x - 5y + 2z = 0,$
   $2x + 2y + 2z = 3.$

4. $x + y + z = 2,$
   $3x - 5y + z = 8,$
   $0x + 0y + 0z = 0.$

5. From the results obtained in Exercise 4, what conclusion is warranted regarding the number of sets of roots belonging to a system of *two* linear equations in *three* variables?

6. $x + y + z = 1,$
   $0x + 0y + 0z = 0,$
   $0x + 0y + 0z = 0.$

7. What do the results obtained in Exercise 6 show regarding the number of sets of roots belonging to *one* equation in *three* variables?

What limit does each of the following expressions approach as n approaches ∞?

8. $\frac{1}{n}$.

9. $\frac{2}{n}$.

10. $\frac{4}{n}$.

11. $\frac{n}{n+1}$.

12. $\frac{n}{n+2}$.

13. $\frac{n+1}{n}$.

HINT. This may be written $1 + \frac{1}{n}$.

14. $\frac{n+2}{n}$.

15. $\frac{\frac{8}{1}}{n}$.

16. $\frac{n(n+4)}{n^2}$.

17. $\frac{n(n+1)(n+2)}{n^3}$.

What limit does each of the following expressions approach as $n \to 0$?

18. $\frac{1}{n}$.

19. $\frac{4}{\frac{1}{n}}$.

20. $\frac{10}{\frac{1}{n}}$.

21. $3\,n^3$.

22. $\frac{n}{n^2}$.

23. $\frac{n(n+1)}{n^2}$.

Find the limit of:

24. $\frac{2-x}{4-x^2}$, when $x \to 2$.

25. $\frac{x^2-5\,x+4}{x^2-1}$, when $x \to 1$.

26. $\frac{x^2+2\,x-3}{x+3}$, when $x \to -3$.

27. $\frac{x+4}{x^3+64}$, when $x \to -4$.

28. $\frac{e^x}{x} = ?$ when $x \to 0$.

29. $\frac{x}{e^x} = ?$ when $x \to 0$.

30. $\frac{n(n-1)!}{n!} = ?$ when $n \to \infty$.

31. $\frac{(n-2)!}{(n-1)!} = ?$ when $n \to \infty$.

32. $\frac{n!}{3\,n(n-2)!} = ?$ when $n \to \infty$.

CHAPTER XXIII

PERMUTATIONS AND COMBINATIONS

167. Permutations. It is often necessary to determine in how many orders a given number of things can be arranged. Suppose we have three things denoted by the letters a, b, and c. They can be arranged in six different orders, as follows:

$a\ b\ c$	$b\ a\ c$	$c\ a\ b$
$a\ c\ b$	$b\ c\ a$	$c\ b\ a$

Each of these arrangements is called a *permutation.*

Suppose we have four things denoted by the letters a, b, c, and d. If we take them two at a time, we may arrange them in twelve different orders, as follows:

$a\ b$	$b\ a$	$c\ a$	$d\ a$
$a\ c$	$b\ c$	$c\ b$	$d\ b$
$a\ d$	$b\ d$	$c\ d$	$d\ c$

Each of these twelve arrangements is also called a permutation.

In general, *each arrangement of all or part of a number of things in a definite order is a permutation.*

If the number of objects is small, the number of permutations possible can be obtained by counting as in the two preceding illustrations. If the number of things is large, counting becomes laborious. It is desirable, therefore, to derive a general formula applicable to problems involving permutations of a large number of things. The derivation of such a formula depends on the following theorem.

168. Fundamental Theorem. *If one act can be performed in* p *ways, and after it is done in one of these ways another act can be performed in* q *ways, then the two can be performed in this order in* pq *ways.*

The truth of this theorem follows when we consider that for one way of doing the first act there are q ways of doing the second. Hence there are, in all, pq ways of doing the two. An obvious generalization of the fundamental theorem is the following:

If one act can be performed in p *ways, then a second act in* q *ways, then a third in* r *ways, and so on, then the number of ways in which they can all be performed is the continued product* pqr ···.

169. The number of permutations of n different things taken r at a time. Let this number be denoted by the symbol ${}_nP_r$. Suppose the things to be arranged be represented by the letters a_1, a_2, a_3, a_4, ···. Then the first place in any group can be filled by the letters in n ways. When this place has been filled, the second place can be filled by any one of the remaining $n-1$ letters in $n-1$ ways; when these two places have been filled, the third place can be filled by any one of the remaining $n-2$ letters in $n-2$ ways.

From these three steps it is clear that the number of possible choices is $n-3$ for the fourth place, $n-4$ for the fifth place, $n-5$ for the sixth place, and so on.

Hence, when the last or rth place is to be filled, $n-(r-1)$ letters remain to be selected from.

Therefore

$$ {}_nP_r = n(n-1)(n-2)(n-3)\cdots(n-r+1). \qquad (1) $$

It can be seen that, if $r = n$ in (1), we obtain

$$ {}_nP_n = n(n-1)(n-2)(n-3)\cdots(n-n+1) = n!. $$

That is,

***The number of permutations of* n *things taken all at a time is factorial* n.**

EXERCISES

1. A room has three doors leading to a hall, and four windows opening outdoors. If all the doors and windows are open, in how many different ways can a bird fly from the hall through the room to the air outside?

2. Two trolley lines run from A to B, three steam lines from B to C, and four motor-bus lines from C to D. In how many different ways can a man travel from A to D?

3. How many different permutations can be made of the letters in the word "house" when taken four at a time?

4. How many integers having three unlike digits can be formed from the numbers 1, 2, 3, 4, 5, 6?

5. In how many ways can five persons take seats together on a bench without sitting twice in the same order?

6. How many signals can be made by flying three flags with seven different flags to select from?

7. In how many ways can four coins fall when tossed in the air?

8. In how many ways can six books on physics and eight on chemistry be arranged on a shelf so that the physics books shall all be together, and the chemistry books also?

9. In how many ways can seven persons be seated at a round table?

HINTS. In any one seating arrangement every one can move to the seat on his left without changing the order. Six such moves can be made with no change of order. Can such a change be carried out with seven persons seated at a straight lunch counter having seven seats?

10. In how many ways can six keys be put on a key ring?

11. How many integers less than 400 can be formed from the digits 1, 2, 3, 4, 5, 6 if none are repeated?

12. Show that ${}_nP_r = \dfrac{n!}{(n-r)!}$.

13. Find n if ${}_nP_4 = 5({}_nP_3)$.

14. Find r if ${}_9P_r = 3({}_8P_r)$.

HINT. $9 \cdot 8 \cdot 7 \cdots (10-r) = 3[8 \cdot 7 \cdot 6 \cdots (9-r)]$

$9 \cdot 8 \cdot 7 \cdots (10-r) = 3[8 \cdot 7 \cdot 6 \cdots (10-r)(9-r)]$.

Whence $9 = 3(9-r)$, etc.

15. Find r if ${}_9P_r = 6({}_7P_r)$.

16. Find n if ${}_nP_6 = 2({}_nP_5)$.

17. Find n if ${}_nP_4 = 12({}_nP_2)$.

18. In how many different ways can four letters be mailed in three different mail boxes?

170. Number of permutations of n things not all different. The letters of the word *consult* can be arranged in 7! or 5040 different ways. The number of permutations of the letters in the word *element*, however, is not 7! because three of the letters are alike. If we affix subscripts 1, 2, and 3 to the *e*'s so as to distinguish one from another, we can arrange these three letters in 3! different ways. Consider, now, any fixed arrangement for the letters as *m e l t e n e*.

This one fixed order, *meltene* giving only one permutation, yields, if the *e*'s are distinguishable, the following six or 3!:

$m\ e_1 l\ t\ e_2 n\ e_3$	$m\ e_2 l\ t\ e_3 n\ e_1$
$m\ e_1 l\ t\ e_3 n\ e_2$	$m\ e_3 l\ t\ e_1 n\ e_2$
$m\ e_2 l\ t\ e_1 n\ e_3$	$m\ e_3 l\ t\ e_2 n\ e_1$

Since this holds for any fixed order of the letters, it follows that the number of permutations in the word *element* is $\frac{7!}{3!}$. In general, the number of permutations of n things r of which are alike, taken all at a time, is $\frac{n!}{r!}$.

Similar reasoning will show that, since s occurs twice and i occurs three times in the word *distinguish*, the number of permutations of its letters, when all are taken at once, is $\frac{11!}{2!\,3!}$.

More generally, the number of permutations of n things of which p, q, and r are alike, when taken all at a time, is

$$\frac{n!}{p!\,q!\,r!}.$$

EXERCISES

1. How many permutations are there of the letters of the word *algebra*?

2. How many permutations are there of the letters of the word *Arkansas*?

3. How many permutations are there of the letters of the word *mathematics*?

4. How many signals can be made by flying three red, two white, and four blue flags one above another?

5. How many numbers, each less than 400, can be made from the digits 1, 2, 3, 4, 5, if none are repeated?

6. In Exercise 5, how many integers can be formed if repetitions are allowed?

7. How many signals can be made in Exercise 4 if the top flag must always be red?

8. In Exercise 4, if the top flag is blue and the bottom one red, how many signals are possible?

9. How many products containing three numbers each may be formed, if only one number is selected from each row and column of

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}?$$

10. Answer the same questions as in Exercise 9, for

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}.$$

171. Combinations. A group of things, not all necessarily different, arranged in any order whatever, is a *combination.*

Thus the letters of two different words, like *restrain* and *strainer*, form but one combination. It follows from the definition that two groups of n objects form two different combinations if a single object in one group is not in the other. Moreover, if we have n things, they form but one combination if taken all at a time in any order whatever. But, for example, if we have five things we can form many combinations of them when we select but three at a time. Thus, if we select three letters at a time from the five letters a, b, c, d, e, we shall have the following ten combinations:

$a\ b\ c$	$a\ b\ e$	$a\ c\ d$	$b\ c\ d$	$b\ d\ e$
$a\ b\ d$	$a\ d\ e$	$a\ c\ e$	$b\ c\ e$	$c\ d\ e$

The important problem here is to determine how many combinations are possible for n things taken r at a time. Consider one combination of r things. This single group of things, when arranged in all possible orders, yields $r!$ permutations. Another combination of r things also gives

$r!$ permutations, and so on. Hence, if ${}_nC_r$ represents the total number of combinations, the total number of permutations is ${}_nC_r r!$.

Therefore $${}_nC_r r! = {}_nP_r. \tag{1}$$

Whence $${}_nC_r = \frac{{}_nP_r}{r!}. \tag{2}$$

PROBLEMS

1. Find the value of ${}_9C_6$ and ${}_{10}C_6$.

2. If ${}_nC_3 = 84$, find n.

3. If ${}_nP_2 = 30$, find n.

4. From ten men, how many different committees of three each can be formed?

5. How many chords can be obtained from eight bells of different pitch, by striking three at a time?

6. How many chords of three notes each can be struck from the black and white keys in one octave on a piano?

7. From nine consonants and four vowels, how many sequences of letters can be formed, each sequence containing three consonants and two vowels?

HINT. If the consonants can be selected in n ways and the vowels in m ways, the two acts can be performed in $m \cdot n$ ways.

8. In how many ways can two men and two women be selected for a game of tennis doubles, having a man and a woman as partners on each side? In how many ways can six men and four women arrange a game of tennis doubles, so that there will be only one man on each side of the net?

9. Show that ${}_nC_r = \dfrac{n!}{r!(n-r)!}$.

HINT. What is the next integer after $n - r + 1$?

10. Show that ${}_nC_r = {}_nC_{n-r}$.

11. Which form in Exercise 10 is the easier to evaluate ${}_nC_r$ or ${}_nC_{n-r}$?

12. In how many ways can ten different books be given to three boys, so that one shall receive five books, one three, and the other two?

13. In how many ways can three prizes be won by two players, if each player may win one or more?

14. How many straight lines are determined by nine points, no three of which are in a straight line?

15. How many tetrahedrons are determined by twelve points, if no four points are in one plane?

16. If five coins are tossed up, in how many ways can they fall?

17. In how many different ways can two dice fall? How many of these ways give 7? How many give 11? What then is the chance that in one throw 7 will turn up? 11?

18. A railway signal has three arms, and each may take four positions. How many different signals can be arranged?

19. Of twenty things how many must be taken at a time to give the largest possible number of combinations?

CHAPTER XXIV

MATHEMATICAL INDUCTION

172. Mathematical induction. The truth of many theorems of algebra has been discovered far in advance of their proof. Often such theorems can be stated in terms of few letters, and their truth can be verified for many numeric values of the letter or letters involved. Even extended numerical verification, however, does not constitute a proof, and ultimately it may become extremely important to establish the truth of a theorem for any value of the letter involved. Sometimes it is sufficient to prove theorems for integral values of the letters used in the theorem. It is often possible to do this with certainty and elegance by a method of proof called *mathematical induction.* The method is most easily shown by an example.

Example

Prove for all values of n that

$$1^2 + 2^2 + 3^2 + \cdots n^2 = \frac{n(n+1)(2n+1)}{6}. \qquad (1)$$

Proof.

If $n = 1$, we have $1 = \frac{1(1+1)(2+1)}{6} = 1.$

If $n = 2$, we have $1 + 4 = \frac{2(2+1)(4+1)}{6} = 5.$

If $n = 3$, we have $1 + 4 + 9 = \frac{3(3+1)(6+1)}{6} = 14.$

If $n = 4$, we have $1 + 4 + 9 + 16 = \frac{4(4+1)(8+1)}{6} = 30.$

If $n = 5$, we have $1 + 4 + 9 + 16 + 25 = \frac{5(5+1)(10+1)}{6} = 55.$

For $n = r = 5$ we may now write

$$1^2 + 2^2 + 3^2 + \cdots r^2 = \frac{r(r+1)(2r+1)}{6}. \qquad (2)$$

Now (2) is known to be true for all values of r from 1 to 5. Hence we can prove it true for $r + 1$, or 6 terms, not by substituting as before but by adding the $(r + 1)$st term to each member of (2) where r is not greater than 5.

$$1^2 + 2^2 + 3^2 \cdots r^2 + (r+1)^2 = \frac{r(r+1)(2r+1)}{6} + (r+1)^2 \qquad (3)$$

$$= (r+1)\frac{[r(2r+1) + 6(r+1)]}{6}$$

$$= (r+1)\frac{[2r^2 + 7r + 6]}{6}$$

$$= (r+1)\frac{(r+2)(2r+3)}{6}$$

$$\text{or} \quad 1^2 + 2^2 + 3^2 + \cdots r^2 + (r+1)^2 = \frac{(r+1)(r+2)(2r+3)}{6}. \qquad (4)$$

Now (4) is the same as (2), except that wherever r occurs in (2), $r + 1$ is present in (4). Hence (1) itself is true for $n = r + 1$, if it is true for $n = r$. But we know it is true for $n = 5$. Hence it is true for $n = 5 + 1$, or 6. But if it is true for $n = 6$, it is true for $n = 6 + 1$, or 7, and so on. Therefore (1) is true for all values of n.

This example shows that an argument by mathematical induction consists of two essential parts:

I. *To show by numeric verification that the formula or principle is true for several integral values of the letter involved, preferably for* n = 1 *to* n = 4, *inclusive.*

II. *To show that, if the formula is true for* $n = r$, *it is true for* $n = r + 1$.

From these two steps it readily follows that the formula is true for all values of n.

173. Necessity of both parts of the proof. It is easy to show that either of the two essential steps of a proof by mathematical induction may hold for a formula, but that the other step may fail.

EXAMPLES

1. (*a*) Suppose $1^2 + 2^2 + 3^2 \cdots n^2 = 3n^2 - 5n + 3$. (1)

If $n = 1$, $1 = 3 - 5 + 3 = 1$.

Or, if $n = 2$, $1 + 4 = 12 - 10 + 3 = 5$.

(*b*) Now let $1^2 + 2^2 + 3^2 \cdots r^2 = 3r^2 - 5r + 3$. (2)

Then

$$1^2 + 2^2 + 3^2 \cdots r^2 + (r+1)^2 = 3r^2 - 5r + 3 + (r+1)^2 \quad (3)$$

$$= 4r^2 - 3r + 4. \quad (4)$$

But step (*b*) fails, for the right member of (4) is *not* the same as the right member of (2) with $r + 1$ substituted for r; that is, $4r^2 - 3r + 4$ is not equal to

$$3(r+1)^2 - 5(r+1) + 3 \text{ or } 3r^2 + r - 2.$$

Therefore, though (1) is true for a few values of r when $n = r$, it cannot be proved true for $n = r + 1$. As a matter of fact (1) fails when $n = 3$.

2. Given

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots n(n+1) = \frac{n}{3}(n+1)(n+2) + 7 \quad (1)$$

Suppose (1) true for $n = r$.

Then

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + 4 \cdot 5 \cdots r(r+1) = \frac{r(r+1)(r+2)}{3} + 7. \quad (2)$$

And

$$1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 \cdots r(r+1) + (r+1)(r+2)$$

$$= \frac{r}{3}(r+1)(r+2) + 7 + (r+1)(r+2) \tag{3}$$

$$= \frac{r(r+1)(r+2)}{3} + \frac{3(r+1)(r+2)}{3} + 7 \tag{4}$$

$$= \frac{(r+1)(r+2)(r+3)}{3} + 7. \tag{5}$$

The right member of (5) is the same as the right member of (2) if r is there replaced by $r+1$. Hence it follows that if (1) is true when $n = r$ it is also true when $n = r+1$. But (1) is not true for any value of n as can be shown by substituting 1, 2, 3, etc. for n.

Theorems requiring proof by mathematical induction are not always series formulas in which step (*b*) is easily carried out by adding the $(r+1)$st term to each member of the formula. Exercises 11 and 19, on pages 351 and 352, and section 174 illustrate the use of wholly different devices which must be employed in order to carry out the second step of a proof by mathematical induction.

NOTE. It is astonishing how far certain formulas will stand numeric verification, yet ultimately fail. It can be proved that no rational algebraic formula for the determination of prime numbers exists, yet many formulas have been discovered which give prime numbers for many consecutive values of n. Those who discovered such formulas often believed them true for all integral values of n. For example, the Chinese, as early as the time of Confucius, believed that n was prime if $2^{n-1} - 1$ was exactly divisible by n. The least value for n for which this theorem is not true is 341. Legendre (1752–1833) gave $2n^2 + 29$ as a formula for primes, and Euler (1707–1783) gave $n^2 + n + 41$. Both of these fail before n reaches 50. Of course each of these men knew that his formula was not true generally. If the student will determine the least value of n for which either formula fails, he will realize that a large number of numeric verifications of a

theorem is very far indeed from a general proof. The necessity of following several numeric verifications by the second essential step in a proof by mathematical induction should then be apparent.

EXERCISES

Prove by mathematical induction that:

1. The sum of the first n integers is $\frac{n}{2}(n+1)$.

2. The sum of the first n odd integers is n^2.

3. The sum of the squares of the first n odd integers is $\frac{n}{3}(2n+1)(2n-1)$.

4. $2^1 + 2^2 + 2^3 + \cdots 2^n = 2(2^n - 1)$.

5. $3 + 6 + 9 + \cdots 3n = \frac{3n}{2}(n+1)$.

6. $1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \cdots n(n+1) = \frac{n}{3}(n+1)(n+2)$.

7. $2 \cdot 5 + 3 \cdot 6 + \cdots (n+1)(n+4) = \frac{n(n+4)(n+5)}{3}$.

8. $1^3 + 2^3 + 3^3 + \cdots n^3 = \frac{n^2}{4}(n+1)^2$.

9. $4^2 + 7^2 + 10^2 + \cdots (3n+1)^2 = \frac{n}{2}(6n^2 + 15n + 11)$.

10. $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} \cdots \frac{1}{n(n+1)} = \frac{n}{n+1}$.

11. Prove that $x^{n+1} - y^{n+1}$ is exactly divisible by $x - y$.

HINT. $x^{n+1} - y^{n+1} = x^{n+1} - xy^n + xy^n - y^{n+1}$
$= x(x^n - y^n) + y^n(x - y)$.

The right member of this identity is divisible by $x - y$ if $x^n - y^n$ is divisible by $x - y$.

12. Prove that $x^{2n} - y^{2n}$ is divisible by $x + y$ if n is a positive integer.

13. Prove that $x^n + y^n$ is divisible by $x + y$ if n is an odd positive integer.

14. Prove that $9^{n+1} - 1$ is exactly divisible by 4.

HINT. Compare $9^{n+1} - 1$ with $x^n - y^n$, and 4 with $x + y$, then study the hint given in Exercise 11.

15. Prove that $3^{2n} - 1$ is exactly divisible by 8.

16. Prove that $9^{n+1} - 1$ is exactly divisible by 8.

17. Prove that $3n^2 + 15n + 6$ is exactly divisible by 6.

18. Prove that $n(n+1)(n+5)$ is exactly divisible by 6.

19. Prove that $(9^{n+1} - 8n - 9) \div 64$ gives an integral quotient for all values of n.

SOLUTION. It is easily shown that the formula holds for $n = 1, 2, 3, 4$, etc.

Hence, if c is a properly chosen integer, we may write:

$$9^{n+1} - 8n - 9 = 64c. \quad (1)$$

Then $$9(9^{n+1} - 8n - 9) = 9 \cdot 64 \cdot c \quad (2)$$

and $$9(9^{n+1} - 8n - 9) + 64n + 64 = 9 \cdot 64 \cdot c + 64n + 64. \quad (3)$$

Whence $$9^{n+2} - 72n - 81 + 64n + 64 = 64(9c + n + 1). \quad (4)$$

Or $$9^{n+2} - 8(n+1) - 9 = 64(9c + n + 1). \quad (5)$$

Since the right member of (5) is divisible by 64, so is the left member. But (5) is the same as (1) if $n + 1$ replaces n, etc.

20. Prove that $3^{2n} - 8n - 1$ is exactly divisible by 64.

21. From the formulas given in Exercises 1 and 2 on the preceding page, derive a formula for the sum of the first n even integers.

22. A pyramid of shot stands on a triangular base having n shots on a side. Show that the number of shots in the pile is $\frac{n(n+1)(n+2)}{6}$. (See Exercise 6, page 351.)

23. A pyramid of shot stands on a square base having n shots on a side. How many shots are there in the pile?

24. The successive intervals between n stones placed on the ground in a straight line are 1 rod, 3 rods, 5 rods, etc. A boy collects the stones one by one and places them in

a box located at the first stone. Show that the total distance he travels is $\frac{n}{3}(n-1)(2n-1)$ rods.

HINT. See Exercise 38, page 283, and section 172.

174. Proof of the binomial theorem. On page 228 is given the proof (for small values of n) that:

$$(a+b)^n = a^n + na^{n-1}b + \frac{n(n-1)}{2!}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{3!}a^{n-3}b^3 \cdots. \quad (1)$$

Then

$$(a+b)^r = a^r + ra^{r-1}b + \frac{r(r-1)}{2!}a^{r-2}b^2 + \frac{r(r-1)(r-2)}{3!}a^{r-3}b^3 + \cdots. \quad (2)$$

Multiplying both members of (2) by $a+b$ gives

$$(a+b)^{r+1} = a^{r+1} + a^rb + ra^rb + ra^{r-1}b^2 + \frac{r(r-1)}{2!}a^{r-1}b^2 + \frac{r(r-1)}{2!}a^{r-2}b^3 + \frac{r(r-1)(r-2)}{3!}a^{r-2}b^3 + \frac{r(r-1)(r-2)}{3!}a^{r-3}b^4 + \cdots \quad (3)$$

$$= a^{r+1} + (r+1)a^rb + \left[r + \frac{r(r-1)}{2!}\right]a^{r-1}b^2 + \left[\frac{r(r-1)}{2!} + \frac{r(r-1)(r-2)}{3!}\right]a^{r-2}b^3 \cdots \quad (4)$$

$$= a^{r+1} + (r+1)a^rb + \frac{2r + r^2 - r}{2!}a^{r-1}b^2 + \frac{3r^2 - 3r + r^3 - 3r^2 + 2r}{3!}a^{r-2}b^3 + \cdots \quad (5)$$

$$= a^{r+1} + (r+1)a^rb + \frac{r(r+1)}{2!}a^{r-1}b^2 + \frac{r(r+1)(r-1)}{3!}a^{r-2}b^3 \cdots. \quad (6)$$

$$(a+b)^{r+1} = a^{r+1} + (r+1)a^r b + \frac{(r+1)[(r+1)-1]}{2!} a^{(r+1)-2}b^2$$
$$+ \frac{(r+1)[(r+1)-1][(r+1)-2]}{3!} a^{(r+1)-3}b^3 \cdots. \qquad (7)$$

Now (7) is the same as (2), with $r+1$ replacing r. Hence if (1) is true when $n = r$, it is true when $n = r+1$. But (1) is true when $n = 3$, hence it is true when $n = 3+1$, or 4, etc. Hence it is true generally.

While the binomial theorem is true for integral, fractional, negative, and irrational exponents, it should be noted that the proof here given demonstrates its truth for integral exponents only.

NOTE. Many theorems can be easily proved by induction which are difficult to establish by other methods. Sometimes the reverse is true. It should be noted, however, that a proof by mathematical induction is rigorous and conclusive. This is not true of the inductive method of science. There we cannot be sure a law holds in every possible case, but only in those cases which we have examined. It is highly probable that Newton's law of gravitation, for example, holds between any two stars in the universe, but it has not been demonstrated. In fact, the inductive method of science is never absolutely demonstrative, but the method of mathematical induction is as rigorous as any other process in mathematics.

CHAPTER XXV

ELEMENTARY STATISTICS

175. Introduction. In many fields of science and business it is frequently necessary to make comparisons between situations which cannot be represented adequately without a very large number of measurements. The tremendous number of such measurements makes direct comparison between two such sets of data out of the question. This fact will be seen immediately if one tries to compare, let us say, the records of length of life of several thousand men, with the same records for an equal number of women. Nevertheless, such comparisons must be made by all life insurance companies, as a basis for determining the premium rates for various classes of people desiring to take out life insurance.

It is therefore necessary to devise some means of summarizing accurately and briefly large masses of figures representing a specific situation. The methods followed in analyzing such sets of figures, and presenting them clearly for comparison, are known collectively as "Statistics." This may be stated more concisely in the following

DEFINITION. *Statistics is the name applied to a group of mathematical and graphical procedures by which the meaning and significance of large masses of observations are studied and made clear.*

In this chapter an attempt will be made to present clearly and simply a few of the fundamental devices used in making a statistical analysis. In order to bring out the

principles involved in each process, very simple examples and exercises will be given. The student should bear in mind, however, that almost all of the problems of statistics in actual practical work involve a great many more measurements than are represented in the problems of this chapter.

176. The frequency distribution. When a set of observations are first recorded they will generally be arranged in no regular order, but will merely occur in the order in which they were set down at the time the measurements were made. The first step in a statistical analysis is to arrange the raw data according to some order. Often the most convenient arrangement is in the order of magnitude. Frequently each value of magnitude will occur more than once. This will lead to the formation of a table known as a *frequency distribution*, as illustrated in the following

EXAMPLE

The following table is a list of the weights, in pounds, of one hundred college freshmen. Make up a frequency distribution for this data.

91.8	127.0	127.0	132.0	136.1	113.9	114.3	122.0	129.0	132.6
122.5	152.2	170.0	185.3	148.0	98.9	103.0	120.0	124.8	147.4
130.8	125.8	123.3	111.8	119.9	128.0	140.0	146.3	144.0	124.5
124.0	109.2	106.7	120.9	121.3	131.4	140.7	112.6	142.1	149.1
118.8	189.9	135.1	115.0	110.8	116.7	120.3	173.4	178.8	135.3
138.7	118.0	124.6	131.0	159.8	137.1	150.7	130.0	163.2	127.7
129.8	116.3	119.2	130.2	143.1	156.3	141.2	165.1	145.1	141.8
139.8	157.9	136.8	128.7	115.8	122.5	126.1	128.2	137.8	151.3
146.0	161.0	167.4	144.3	153.3	126.2	134.0	139.1	149.9	154.0
131.9	129.9	126.8	133.0	138.1	142.6	134.2	155.1	133.4	134.3

An inspection of the data above will show that there are nearly one hundred different weights represented.

Experience has shown that many types of data are adequately summarized in a frequency distribution if approximately twenty divisions of the magnitude are taken. That is, in the above table, since the minimum weight is slightly over 90 pounds and the maximum is slightly less than 190 pounds, we will be able to form twenty weight groups by taking the various weights in groups of five pounds each. Thus the weights will be grouped as follows:

Over 90 pounds and less than 95 pounds
95 pounds and less than 100 pounds
100 pounds and less than 105 pounds
and so forth.

When the given weights are rearranged in this manner, the following frequency distribution will be obtained.

Weight Group	Frequency
Over 90, less than 95 pounds	1
95, less than 100 pounds	1
100, less than 105 pounds	1
105, less than 110 pounds	2
110, less than 115 pounds	5
115, less than 120 pounds	8
120, less than 125 pounds	12
125, less than 130 pounds	13
130, less than 135 pounds	13
135, less than 140 pounds	10
140, less than 145 pounds	9
145, less than 150 pounds	7
150, less than 155 pounds	5
155, less than 160 pounds	4
160, less than 165 pounds	2
165, less than 170 pounds	2
170, less than 175 pounds	2
175, less than 180 pounds	1
180, less than 185 pounds	0
185, less than 190 pounds	2

This rearrangement of the data immediately shows us a significant characteristic of this group of measurements, namely, the tendency of the frequencies to group themselves about a central point in the range of weight groups, at about 130 pounds. This *central tendency* is a common characteristic of certain physical measurements.

EXERCISES

Make up frequency distribution tables for the following:

1. 2, 5, 0, 4, 1, 2, 3, 2, 1, 2, 3, 4, 5.

2. 0, 2, 2, 4, 8, 2, 4, 6, 4, 2, 4, 6, 4.

3. 5, 15, 10, 0, 5, 5, 15, 5, 10, 10, 5.

4. 10, 0, 20, 10, 20, 40, 30, 40, 60, 50, 40, 80, 40, 30.

5. 3, 9, 0, 6, 9, 6, 3, 6, 3, 6, 6, 6, 3, 3, 6.

6. 0, 12, 8, 8, 4, 4, 16, 20, 4, 16, 12, 8, 4, 8, 8, 4, 12, 0.

177. Mean, median, and mode. In section 176 we have seen that certain classes of measurements have a tendency to be grouped about a central value. This tendency was called the *central tendency*. In cases where this central tendency is found it is often convenient to possess a means of measuring it. The three most commonly used measures of central tendency are the *mean*, the *mode*, and the *median*.

The *mean* of a group of readings is the *arithmetic average* of the group. It is found by dividing the sum of all the individual readings by the total number of readings in the group.

Thus in the case of the weights of the hundred freshmen the total of all the weights given in the table on page 356 is 13,492.6 pounds. The weight of all the freshmen divided by the total number of freshmen, 100, gives us the mean for this set of measurements, which is 134.926 pounds.

NOTE. A shorter method of obtaining the mean is to use the frequency distribution table as a basis for computation. The assumption is made that all the individual measurements which are grouped together have the same magnitude, which is the mid-point of the group. Thus it is assumed that all the individuals weighing between 130 and 135 pounds have the same weight and that this weight is the mid-point of the group, or 132.5 pounds.

The total weight of all the freshmen will then be the sum of all the products obtained by multiplying the mid-point of each weight group by the frequency of that weight group. This total figure may then be divided by the total frequency to get the mean weight.

While this result is not quite as accurate as the result obtained from the raw data, it is usually close enough for all practical work, provided there are a sufficient number of magnitude groups in the frequency distribution. This is one of the reasons why it is desirable to take between 15 and 20 different groups in making up a frequency distribution.

We may state the expression for the mean as follows:

$$\textit{Mean} = \frac{\textit{Sum of all the measurements}}{\textit{Number of measurements}}.$$

The *mode* is the term applied to that measurement or group which has the greatest frequency. Thus in the example on page 357 we find from the frequency distribution that the groups 125 to 130 pounds and 130 to 135 pounds each have a frequency of 13. These two groups taken together may be considered as the mode, or the mode may be considered to lie between the two groups, at 130 pounds.

The *median* is that value which divides a set of measurements in two equal parts, one part consisting of all the values smaller in magnitude than the median value and the other part consisting of values larger than the median value.

Thus, from a study of the original data in the example, we find that the value halfway between 132.0 and 132.6 pounds will divide the set of measurements into two groups of fifty values each. This median value will be in this case 132.3 pounds.

NOTE. As in the case of the mean there is a shorter method of getting the median value based on the frequency distribution. This consists in adding down the frequency column until the addition of the frequency for the next group would cause the sum to exceed half the total frequency. Then, by a process of interpolation in the next group, the median value may be found.

Thus in the example on page 357 the sum of the frequencies of the smallest eight groups gives a total of 43. The addition of the frequency of the ninth group would make the total 56, which is over half the total frequency. Since 50 is half of the total frequency, we must include 7/13 of the ninth group. Multiplying the weight interval, 5 pounds, by $\frac{7}{13}$, and adding the result to the upper limit of the first eight weight groups, we find that the median value will correspond to 132.7 pounds.

As in the case of the mean this value is only an approximation, but is accurate enough for all practical uses.

The method employed in finding the median can be extended to finding other values which divide the series of observations into any proportion desired. If the value divides the series into sections $\frac{1}{4}$ and $\frac{3}{4}$ of the whole, the value is called a *quartile* instead of a median. If the value divides the total into proportions expressed in percentages, that value is called a *percentile*. Thus we speak of the 10-percentile, or the 30-percentile. Here the 50-percentile would be the median.

EXERCISES

Find the mean, the median, and the mode in the following exercises:

1. 8, 12, 15, 4, 8, 9, 6, 3.
2. 4, 15, 7, 12, 13, 12, 15, 4, 8, 10.
3. 20 in., 42 in., 53 in., 41 in., 38 in.
4. 102 lb., 95 lb., 97 lb., 98 lb., 100 lb.
5. 60 in., 64 in., 70 in., 72 in., 68 in., 69 in., 74 in., 71 in.

6. 200 cm., 180 cm., 190 cm., 185 cm., 190 cm., 200 cm.

7. 68, 65, 70, 67, 70, 60, 61, 66, 82, 90, 68, 92, 70, 75, 67.

8. The following are the barometric readings for a day:

Hour	Inches	Hour	Inches
8 A.M.	29.35	2 P.M.	29.20
9 A.M.	29.45	3 P.M.	28.80
10 A.M.	29.20	4 P.M.	28.70
11 A.M.	29.80	5 P.M.	29.50
Noon	30.24	6 P.M.	29.80
1 P.M.	29.65	7 P.M.	30.20

Find the mean of the above readings.

9. A certain business concern made sales amounting to the following sums:

Jan.	$510	July	$800
Feb.	600	Aug.	750
Mar.	590	Sept.	710
Apr.	630	Oct.	710
May	610	Nov.	750
June	750	Dec.	800

Compute the mean monthly sales for this firm.

10. The monthly rainfall for the City of New York in 1923 was:

Month	Inches	Month	Inches
Jan.	5.97	July	4.68
Feb.	2.33	Aug.	1.12
Mar.	4.08	Sept.	2.34
Apr.	2.07	Oct.	4.65
May	1.58	Nov.	2.26
June	1.86	Dec.	3.78

Compute the mean monthly rainfall.

11. The following is the frequency distribution for the height, in inches, of the members of a certain college gymnasium class:

Height in Inches	Frequency
61	2
62	10
63	11
64	38
65	57
66	93
67	106
68	126
69	109
70	87
71	75
72	23
73	9
74	4

Compute the mean, mode, and median of the above measurements.

Hint. Use the method outlined in the note on page 360.

12. The distribution of ages of graduates from the eighth grade is as follows:

Age	Frequency
11	0
12	7
13	45
14	186
15	114
16	61
17	8
18	0

Find the mean, the mode, and the median for these ages.

13. The following frequency distribution is for the scores or grades made by students in an examination:

Grade Interval	Frequency
0 to 1.99	1
2 to 3.99	1
4 to 5.99	1
6 to 7.99	0
8 to 9.99	1
10 to 11.99	4
12 to 13.99	2
14 to 15.99	0
16 to 17.99	0
18 to 19.99	2

Find the mean, mode, and median for these grades.

Hint. The mid-point of the first interval is 1.

14. Compute the mean, the median, and the mode for the following frequency distribution:

Per Cent Grade Interval	Frequency
0 to 9.9	50
10 to 19.9	50
20 to 29.9	100
30 to 39.9	200
40 to 49.9	600
50 to 59.9	400
60 to 69.9	200
70 to 79.9	100
80 to 89.9	50
90 to 99.9	50

Hint. The mid-point of the first interval is 5.

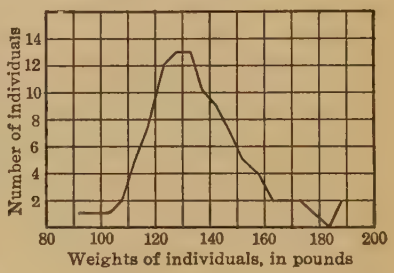
Note. It is not possible to measure in specific units the mental abilities of a student as we may measure his height and weight. By appropriate tests, however, we can measure the responses of a group of students and arrange them in the order of their mental power. The median of such a group can then be determined and its properties compared with other similar medians. In such ratings the arithmetic average is less useful than the median for comparative purposes.

178. Graph of a frequency distribution. As an aid to studying and interpreting frequency distributions it is sometimes desirable to represent them by means of a curve. Such a graph is shown in the following

Example

Graph the frequency distribution of the weights of one hundred college freshmen as given on page 357.

Solution. Along the horizontal axis lay off weights from 80 pounds to 200 pounds. Along the vertical axis lay off intervals up to 15. Now locate a point above the mid-point of each weight interval, at the height corresponding to the correct frequency. Connect these points with a broken line. This will give the adjacent graph. The symmetrical arrangement of the weight groups is at once evident from this graph.



EXERCISES

Graph the following:

1.

Per Cent Grade	Frequency
0	2
5	8
10	15
15	17
20	16
25	12
30	8
35	4
40	2
45	1

2.

Per Cent Grade	Frequency
0	0
5	1
10	3
15	4
20	7
25	6
30	12
35	21
40	16
45	17
50	14
55	10
60	5
65	3
70	1

3. Below is given the distribution for the time required by 100 pupils to complete eight school grades:

Years Required	Number of Pupils
4	1
5	2
6	9
7	21
8	33
9	22
10	9
11	2
12	1

4. This table shows the number of ribs in 26 sample leaves taken from the same birch tree:

Number of Ribs	Number of Leaves
13	0
14	0
15	1
16	4
17	7
18	9
19	4
20	1

5. The frequency distribution of the number of grains of oats per plant is given in this table:

Grains per Plant	Frequency
1	3
2	50
3	106
4	109
5	80
6	42
7	7
8	2
9	1

6. The heights of a very large number of boys $11\frac{1}{2}$ years old were measured. The figures thus obtained were then set down in terms of percentages of the total number examined, as follows:

Height in Centimeters	Per Cent Frequency
109 to 112	0.2
113 to 116	0.2
117 to 120	0.4
121 to 124	2.5
125 to 128	9.1
129 to 132	18.1
133 to 136	25.5
137 to 140	22.3
141 to 144	13.9
145 to 148	5.3
149 to 152	1.9
153 to 156	0.4
157 to 160	0.2

179. Deviation and average deviation. We have already seen the importance of having a measure of central tendency in the form of the mean, the median, and the mode. In many cases the measurements in any set may be very

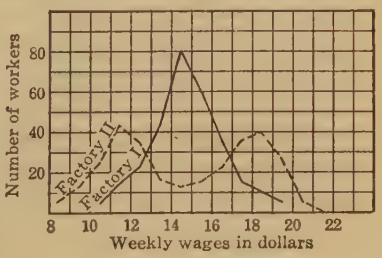
widely scattered about the central value, and yet have a mean identical with that of a group of measurements which are very closely bunched. This condition is illustrated in the following

EXAMPLE

Compute the mean for the following frequency distributions of the wages in two factories:

FACTORY I		FACTORY II	
WAGE	NUMBER	WAGE	NUMBER
$10.50	6	$8.50	5
11.50	14	9.50	12
12.50	22	10.50	25
13.50	45	11.50	44
14.50	80	12.50	36
15.50	60	13.50	18
16.50	35	14.50	12
17.50	15	15.50	15
18.50	10	16.50	22
19.50	5	17.50	36
	Total, 292	18.50	40
		19.50	28
		20.50	6
		21.50	1
			Total, 300

The mean should be found from these figures by multiplying each wage by its respective frequency, adding all these products together, and dividing by the total frequency. The mean for Factory I is $14.78, while the mean wage for Factory II is $14.83. Judging purely by the value of the mean found in each case



we would be led to the belief that the two factories were similar in their wage situation. This is far from true, as a glance at the graphs of the above frequency distributions will show.

The graph for Factory II shows quite the reverse of a central tendency. The need for a measure of this opposite tendency, or *dispersion* as it is called, is evident. Such a measure has been worked out and is known as the *deviation*. This is defined as follows:

DEFINITION. ***The difference between any measurement and the mean of the series of which that measurement is a part is called a deviation.***

This can be stated in the following formula:

$$\textit{Deviation} = \textit{Individual measurement} - \textit{Mean}.$$

From this formula it can be seen that the deviation of values less in magnitude than the mean will be negative in sign, and values greater than the mean will have a positive deviation.

In the preceding example the deviation of the \$10.50 wage in Factory I is $\$10.50 - \$14.78 = -\$4.28$, and the deviation of the \$18.50 wage in Factory II is $\$18.50 - \$14.83 = \$3.67$.

The deviation is representative of only one individual measurement or observation in the series, and there will be as many different deviations as there are measurements. In order to get a number which will represent the dispersion of the series as a whole, we use the *average deviation*, or *A. D.* as it is sometimes abbreviated.

The average deviation is obtained by taking the arithmetic average or the mean of all the deviations from the average or mean of the series. The student should bear in mind that there are as many deviations as there are original

measurements and that in computing the *A. D.* the deviation must be multiplied by the frequency corresponding to that group or interval before it is added together with other similar products to form a total.

This procedure is illustrated in the following

Example

Find the average deviation for the wages of Factory I in the example on page 367.

Solution. The mean of the wages here is \$14.78. Subtracting the mean from each of the wage values and placing the difference in the third column we get:

Wage	Frequency	Deviation	Frequency × Deviation
\$10.50	6	− \$4.28	\$25.68
11.50	14	− 3.28	45.92
12.50	22	− 2.28	50.16
13.50	45	− 1.28	57.60
14.50	80	− .28	22.40
15.50	60	.72	43.20
16.50	35	1.72	60.20
17.50	15	2.72	40.80
18.50	10	3.72	37.20
19.50	5	4.72	23.60
	Total, 292		Total, 406.76

In the fourth column above is recorded the product of each deviation and the corresponding frequency. In this case, where the average deviation is being computed, the algebraic sign of the individual deviations is disregarded, and all signs are considered positive. The average deviation is now obtained by dividing the total of the fourth column by the total of the second column, or total frequency. This gives us the quotient

$$\frac{\$406.76}{292} = \$1.39 = \text{A. D.}$$

The average deviation for the wages of Factory II computed in the same manner is \$3.08. That is, the wages in Factory II are more than twice as widely scattered about the mean as those in Factory I.

EXERCISES

Find the mean and the average deviation for each of the following series of measurements:

1. 3, 5, 12, 7, 8, 2, 4, 9, 10, 6, 5, 4.

2. 2, 8, 0, 6, 4, 5, 12, 8, 0, 4, 6, 8.

3. 5, 0, 10, 5, 20, 35, 20, 25, 15, 25.

4. 4, 6, 4, 12, 20, 50, 10, 24, 6, 8, 10.

5. 20, 50, 25, 40, 60, 40, 45, 44.

6.

Age	Frequency
0 yr.	5
10 yr.	4
20 yr.	6
30 yr.	10
40 yr.	12
50 yr.	3

7.

Length	Frequency
100 in.	2
80 in.	15
60 in.	9
40 in.	4
20 in.	1
0 in.	2

180. Standard deviation. For most approximate work the average deviation is as exact as the accuracy of the original data will warrant. There are cases, however, when a more accurate measure of deviation is required. This need is supplied by the *standard deviation*. The standard deviation is calculated by taking the deviations found as shown for Factory I in the example on page 369. Each deviation is then squared and all such squares are added together. This total of the squared deviations is then divided by the total frequency. This figure is the mean squared deviation. The standard deviation is then found by taking the square root of the mean squared deviation.

The formula for the standard deviation is

$$\sigma = \sqrt{\frac{\textit{Sum of (deviations)}^2}{\textit{Total frequency}}}$$

The symbol for the standard deviation is σ, the Greek letter sigma.

181. Probable error. The principles outlined in the preceding sections find wide application to problems which arise in economics, sociology, and education. Widely extended, with certain additional principles of probability, they form the basis on which the gigantic business of life insurance is built. The natural sciences, especially astronomy, physics, and chemistry find constant applications of the theory of statistics. Two specific instances will be given.

First let us assume that we are faced with a problem of a scientific nature requiring repeated measurements of the same quantity. Since every measurement is subject to error, the arithmetic mean of all the measurements is rarely the absolutely correct value. Frequently it is impossible to determine the correct value. It then becomes desirable to determine the probable error of the mean, and in certain cases of each individual measurement in addition. The process by which this is done is known as the method of least squares.

By a method analogous to that employed in computing the average deviation and the standard deviation, the measurements above may be analyzed to give the *error*, the *average error*, and the *probable error*.

NOTE. Another formula sometimes given for σ is

$$\sigma = \sqrt{\frac{\text{Sum of (each score)}^2}{\text{Number of scores}} - (\text{Mean})^2}.$$

By means of this formula it is possible to compute a standard deviation directly from the raw scores and their mean, without finding the individual deviations of the scores.

It is interesting to note at this point that about 68 per cent of the scores in a set of readings with a normal frequency distribution lie between the two points (mean $-\sigma$) and (mean $+\sigma$).

182. The coefficient of correlation. One of the most useful and important values for the comparison of certain classes of measurements is the *coefficient of correlation.* This coefficient expresses the likelihood of a variation in one measured characteristic being accompanied by a corresponding variation in another characteristic of the group of objects under consideration. The coefficient may range in value from -1 to $+1$. The value 1 indicates that if one characteristic is known the other characteristic may be predicted with certainty without further measurements or study. Thus if in a group of individuals the correlation of weight to height is 1, we would be safe in saying that the tallest person would also be the heaviest, and vice versa, that the lightest person would be the shortest. A negative sign before the coefficient of correlation indicates that the relation between the two characteristics is inverse; that is, the person or thing having the greatest amount of one attribute will tend to have the least amount of the other characteristic. Thus the correlation between the weight of an object and its distance above the earth's surface would be negative in sign.

CHAPTER XXVI

MISCELLANEOUS TOPICS

183. Detached coefficients. A term is said to be *rational* if it may be obtained from unity and the letters involved by means of the four fundamental operations without the extraction of any root.

A term is *integral* if it has no literal denominator and the exponent of each factor is a *positive* number or zero.

An integral expression may not be rational. Nor is every rational expression integral.

Thus, $\frac{x^2}{9}+\sqrt{x}+6$ is integral but not rational, while $\frac{x^2}{9}+\frac{1}{x}+6$ is rational but not integral.

The *degree* of a rational integral term is the sum of the exponents of the letters in the term.

Thus, x^2y and $x^3y^5z^2$ are of the third degree and the tenth degree, respectively.

An algebraic expression is *rational and integral* if its terms are rational and integral.

A rational integral expression is *homogeneous* if its terms are all of the same degree.

For example the expression $x^2+2xy+y^2$ is homogeneous, while the expression $x^3+2x^2y+y^2$ is not homogeneous.

In the multiplication or division of polynomials which involve but one letter or which are homogeneous in two letters much labor can be saved by using the coefficients only.

Examples

1. Multiply $2x^4 + 3x^3 - 2$ by $3x^2 - 5x + 1$.

Solution. Since x^2 and x are missing in the first expression, their coefficients are zero. Inserting $0\,x^2$ and $0\,x$ and detaching coefficients, the multiplication is as follows:

$$\begin{array}{r}
2 + 3 + 0 + 0 - 2 \\
3 - 5 + 1 \\
\hline
6 + 9 + 0 + 0 - 6 \\
-10 - 15 - 0 - 0 + 10 \\
\qquad\qquad 2 + 3 + 0 + 0 - 2 \\
\hline
6 - 1 - 13 + 3 - 6 + 10 - 2
\end{array}$$

Supplying the powers of x, we obtain as the product the complete polynomial expression

$$6x^6 - x^5 - 13x^4 + 3x^3 - 6x^2 + 10x - 2.$$

2. Divide $6x^5 + 13x^4y - 19x^3y^2 - 37x^2y^3 + 17xy^4 + 20y^5$ by $3x^2 + 2xy - 5y^2$.

Solution.

$$\begin{array}{r|l}
6 + 13 - 19 - 37 + 17 + 20 & 3 + 2 - 5 \\
\cline{2-2}
6 + 4 - 10 & 2 + 3 - 5 - 4 \\
\cline{1-1}
9 - 9 - 37 & \\
9 + 6 - 15 & \\
\cline{1-1}
-15 - 22 + 17 & \\
-15 - 10 + 25 & \\
\cline{1-1}
-12 - 8 + 20 & \\
-12 - 8 + 20 & \\
\cline{1-1}
\end{array}$$

Therefore the quotient is $2x^3 + 3x^2y - 5xy^2 - 4y^3$.

In both multiplication and division by detached coefficients, zero must be supplied for the coefficient of any missing term. If this is done a great deal of confusion and many errors will be avoided.

EXERCISES

Use detached coefficients to perform the indicated operation:

1. $(2x^2 - 3x + 5)(3x + 1)$.
2. $(x^2 + 2xy + y^2)(x^2 - 2xy + y^2)$.
3. $(m^2 + mn + n^2)(m^2 - mn + n^2)$.
4. $(6x^2 + x - 2) \div (2x - 1)$.
5. $(15x^2 - 42 + 53x) \div (5x + 21)$.
6. $(2x^2 + 5xy - 7y^2)(3x^2 - 2xy + 6y^2)$.
7. $(2x^2y + 3y^3 - 10x^3 + 4xy^2)(3x^3 - 10x^2y + 25xy^2 - y^3)$.
8. $(9x^4 + 8x^2 + 4) \div (3x^2 + 2x + 2)$.
9. $(24a^4 - 6a^3b + 19a^2b^2 + 16ab^3 - 21b^4) \div (2a^2 - ab + 3b^2)$.
10. $(3x^4 - 7x^3 - x - 10x^{-1} + 15) \div (x^2 - 3x + 2)$.
11. $(4x^4 + 16x^3 + 5x^2 - 25x - 4 - 6x^{-1} + 10x^{-2}) \div (2x^2 + 3x - 5)$.
12. $(3x^2 - 2x + 5x^{\frac{1}{2}} + 3)(2x + 3x^{\frac{1}{2}} + 2)$.
13. $(15x + 7x^{\frac{5}{6}} - 2x^{\frac{2}{3}}) \div (3x^{\frac{1}{2}} + 2x^{\frac{1}{3}})$.
14. $(x^5 + 4x^4y + 2x^3y^2 - 4x^2y^3 + xy^4 + y^5) \div (x^2 + 3xy + y^2)$.
15. Which expressions in the preceding exercises are (*a*) not integral? (*b*) not rational?

NOTE. It is interesting to observe that our ordinary decimal notation really involves the use of detached coefficients. The number 423, for instance, is an abbreviated way of writing $4 \cdot 10^2 + 2 \cdot 10 + 3$. In fact, the various digits in any number in the decimal form are the detached coefficients of some power of 10.

184. Synthetic Division. This method of division abbreviates the work of long division and may be used where

the divisor is a binomial, and it gives, therefore, a short method of determining whether or not a given polynomial contains that binomial as a factor.

EXAMPLES

1. Divide $4x^3 - 8x^2 + x + 6$ by $x + 1$.

Solution. By long division we have

$$
\begin{array}{r}
4x^2 - 12x + 13 \\
x+1 \overline{)\,4x^3 - 8x^2 + x + 6} \\
\underline{4x^3 + 4x^2} \\
-12x^2 + x \\
\underline{-12x^2 - 12x} \\
13x + 6 \\
\underline{13x + 13} \\
-7, \text{ Remainder}
\end{array}
$$

The preceding division can be shortened by omitting the letters and arranging the work as was done in Example 2, page 374. However, the process just given can be shortened still further. We may also omit the first number of each partial product, since it is merely a repetition of the number just above it. Thus we obtain

$$
\begin{array}{r}
4 - 12 + 13 \\
1+1 \overline{)\,4 - 8 + 1 + 6} \\
\underline{+4} \\
-12 \\
\underline{-12} \\
+13 \\
\underline{+13} \\
-7, \text{ Remainder}
\end{array}
$$

If we change the plus sign before the 1 in the second term of the divisor to a minus sign, we will change all the signs in the partial products. We may then *add* the partial products thus found to the dividend instead of subtracting them. (This

change of sign is not an actual necessity, but it is a great convenience in practical work.) Then bringing the figures into horizontal lines and using only the second term of the divisor with its sign changed, we have a further abbreviation of the process:

$$\begin{array}{r|rrrr} -1 & 4 & -8 & +1 & +6 \\ & & -4 & +12 & -13 \\ \hline & 4 & -12 & +13 & -7 \end{array}$$

Here we have all the essential work for the complete division of $4x^3 - 8x^2 + x + 6$ by $x + 1$, for the figures on the lower line, $4, -12, 13$, preceding the remainder, -7, are the coefficients of the partial quotient $4x^2 - 12x + 13$.

2. Divide $2x^4 - 15x^3 + 27x^2 - 13x + 15$ by $x - 5$.

Solution.

$$\begin{array}{r|rrrr} 5 & 2 & -15 & +27 & -13 & +15 \\ & & +10 & -25 & +10 & -15 \\ \hline & 2 & -5 & +2 & -3 & \end{array}$$

Therefore the quotient is $2x^3 - 5x^2 + 2x - 3$.

Since the remainder is zero, $x - 5$ is a factor of the dividend and the other factor is the quotient $2x^3 - 5x^2 + 2x - 3$.

The last step of Example 1 and the solution of Example 2 above illustrate the method of *synthetic division.*

EXERCISES

Divide by synthetic division:

1. $x^2 - 3x - 10$ by $x - 5$.
2. $x^2 + x - 12$ by $x + 4$.
3. $x^2 + 6x + 5$ by $x + 1$.
4. $x^2 - 3x - 28$ by $x - 7$.
5. $x^2 - 14x + 24$ by $x - 2$.
6. $2x^2 + 5x - 3$ by $2x - 1$.

HINT. $(2x^2 + 5x - 3) \div (2x - 1) = 2(x^2 + \frac{5}{2}x - \frac{3}{2}) \div 2(x - \frac{1}{2})$
$= (x^2 + \frac{5}{2}x - \frac{3}{2}) \div (x - \frac{1}{2})$.

7. $3x^2 - 10x - 8$ by $3x + 2$.

8. $6x^2 - x - 2$ by $2x + 1$.

9. $5x^2 + 24x - 5$ by $x + 5$.

10. $21x^2 - 22x + 5$ by $7x - 5$.

11. $3x^2 - 13x - 10$ by $x - 5$.

12. $4x^2 - 3x - 7$ by $x + 1$.

13. $x^3 + 3x^2 - 13x + 6$ by $x - 2$.

14. $2x^3 + 3x^2 - 10x - 15$ by $2x + 3$.

15. $bx^2 + (ab - c)x - ac$ by $x + a$.

16. $bx^3 - (7b + a)x^2 + (7a + c)x - 7c$ by $x - 7$.

17. $x^3 + 1$ by $x + 1$.

18. $12x^2 + x - 35$ by $4x + 7$.

19. $3x^3 - 25x + 6$ by $x + 3$.

20. $adx^3 + (bd - ae)x^2 - (be - cd)x - ce$ by $dx - e$.

Using synthetic division, find the remainder in Exercises 21, 23, 25, and 27:

21. $(x^2 + 2x - 3) \div (x - a)$.

22. Substitute a for x in $x^2 + 2x - 3$ and compare the result with the remainder obtained in Exercise 21.

23. $(x^2 + ax + n) \div (x - b)$.

24. Put b for x in $x^2 + ax + n$ and again compare results.

25. $(x^3 + ax^2 + bx + c) \div (x - p)$.

26. Compare the remainder in Exercise 25 with the result obtained by putting p in place of x in the dividend.

27. $(x^3 + 3x^2 - 2) \div (x - 2)$.

28. Substitute 2 for x in $x^3 + 3x^2 - 2$ and compare the result with the remainder obtained in Exercise 27.

29. What general conclusion is to be drawn from the results of Exercises 21 to 28?

NOTE. The approximate solution of equations of the third and higher degrees, having numeric coefficients, was a problem to which Newton devoted considerable attention. Little progress in this line was made from his time until 1819, when William George Horner (1786–1837), a teacher in Bath, England, published a method of solving equations by synthetic division. His procedure was in essence very similar to Newton's, and its element of originality lay in the very compact and elegant form in which he arranged the numeric work. In the ninety years which have intervened since its publication Horner's method has been improved but little, which is rather remarkable as Horner did not have the advantage of a university training and was by no means a great mathematician.

185. Factors involving imaginaries. After studying radicals we could have enlarged our previous notion of a factor and, with certain limitations, employed radicals among the terms of a factor. Now in a similar manner, with certain restrictions, we may extend our notion of a factor still further and use imaginary numbers as coefficients or as terms in a factor. For example, $4x^2 + 1$ may hereafter be regarded as factorable into complex factors, for $4x^2 + 1 = 4x^2 - (-1) = (2x + \sqrt{-1})(2x - \sqrt{-1})$.

Similarly, $x^2 + 4 = x^2 - (-4)$

$$= (x + 2\sqrt{-1})(x - 2\sqrt{-1}),$$

$$9n^2 + 5 = 9n^2 - (-5)$$

$$= (3n + \sqrt{-5})(3n - \sqrt{-5}),$$

and $y^2 + 2y + 2 = y^2 + 2y + 1 - (-1)$

$$= (y + 1 + \sqrt{-1})(y + 1 - \sqrt{-1}).$$

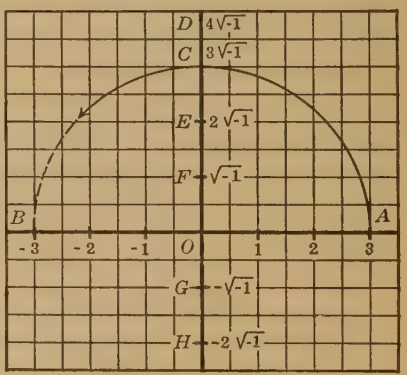
Further, $x^3 + 8 = (x + 2)(x^2 - 2x + 4)$. Hitherto the trinomial $x^2 - 2x + 4$ has been regarded as prime, but the student can easily prove that $x^2 - 2x + 4$ is equal to the product $(x - 1 + \sqrt{-3})(x - 1 - \sqrt{-3})$. Similarly, $x^3 + 8$ has three factors, $x + 2$, $x - 1 + \sqrt{-3}$, and $x - 1 - \sqrt{-3}$.

EXERCISES

Find the complex factors of the following:

1. $x^2 + 1$.
2. $m^2 + 2$.
3. $4a^2 + 9$.
4. $25x^2 + 11$.
5. $x^2 + y^2$.
6. $8x^3 + 27$.
7. $n^2 - 8n + 21$.
8. $x^2 + 7x - 12$.
9. $z^2 + 6z + 10$.
10. $4x^2 + 12x + 15$.
11. $x^2 + 100ix - 25$.
12. $x^2 + x\sqrt{-36} - 9$.

186. Graphical interpretation of pure imaginaries. In our previous graphical work a positive number and a numerically equal negative number, as $+3$ and -3, were represented by equal distances measured in opposite directions on a straight line from O as an origin, such as OA and OB of the adjacent figure. Now multiplying $+3$ by -1 gives $-1 \cdot (+3)$ or -3. Hence, we may regard -1 as an operator (rotor) which turns OA about O as a center in the direction of the arrow (counterclockwise) into the position OB, or



through two right angles (180°). This rotation through 180° corresponds to a change from a positive to a negative number of the same absolute value.

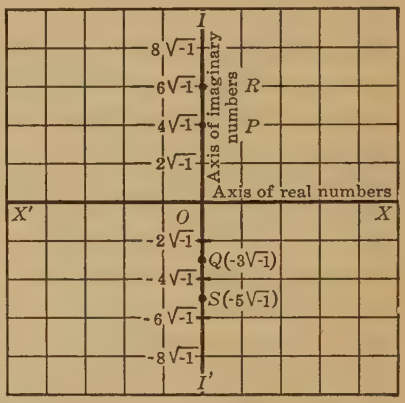
The preceding illustration suggests a method of locating a number such as $\sqrt{-1} \cdot (+3)$ which is in strict conformity with our previous graphical work, for $-1 = \sqrt{-1} \cdot \sqrt{-1}$.

Since multiplying a number by -1 produces the same effect as multiplying twice in succession by $\sqrt{-1}$, then multiplying by $\sqrt{-1}$ once may be regarded as producing a rotation, as in the figure on page 380, of but one right angle (90°). Thus in the above figure, OA, or $+3$ multiplied by $\sqrt{-1}$, would be turned to the position OC. Hence the point C is said to correspond to the number $3 \cdot \sqrt{-1}$. Similarly, point F corresponds to $1 \cdot \sqrt{-1}$ and point D to $4 \cdot \sqrt{-1}$. Then OG, being measured in a direction opposite to OF, OE, OC, and OD, would, from previous graphical work, correspond to $-\sqrt{-1}$. In like manner, OH corresponds to $-2\sqrt{-1}$. This last number, however, may be reached differently, for multiplying 2 by $\sqrt{-1}$ three times gives $-2\sqrt{-1}$. Hence these three successive multiplications by $\sqrt{-1}$ may be regarded as producing a counterclockwise rotation through three right angles, which would locate the point corresponding to $-2\sqrt{-1}$ at H as before. If the student takes up the study of the theory of alternating currents, he will find that a rotation through any angle can be represented as multiplication by a suitable complex number.

Therefore, a pure imaginary number such as $b\sqrt{-1}$ (see diagram, page 380) is represented by a point on an axis perpendicular to the axis of real numbers. This point will be located b units in the direction of OD if b is positive and in the direction of OH if b is negative. The new axis will hereafter be called the imaginary or I-axis.

Example

Locate the points corresponding to the following imaginary numbers: P, $4\sqrt{-1}$; Q, $-3\sqrt{-1}$; R, $6\sqrt{-1}$; and S, $-5\sqrt{-1}$.



Point P, which represents $4\sqrt{-1}$, is located upward from the origin a distance of 4 units on the imaginary axis, since the number is positive; point Q is located downward from O a distance of 3 units on the imaginary axis. The other points are located in a similar manner.

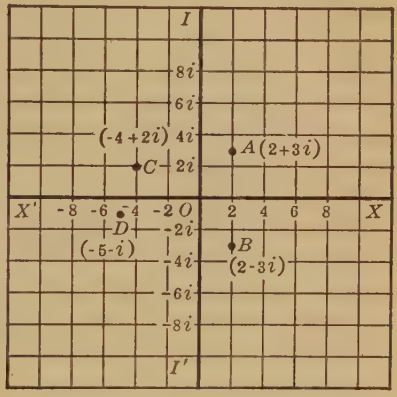
EXERCISES

Represent graphically the points for the following imaginary numbers:

1. $2\sqrt{-1}$.	6. $-2i$.	11. $\sqrt{3}\cdot\sqrt{-3}$.
2. $-5\sqrt{-1}$.	7. $+\sqrt{3}\,i$.	12. $\sqrt{-32}$.
3. $-\sqrt{-16}$.	8. $-\sqrt{5}\,i$.	13. $-\frac{1}{4}\sqrt{-36}$.
4. $\sqrt{-25}$.	9. $-2\sqrt{-9}$.	14. $\sqrt{-5}$.
5. $3i$.	10. $2\sqrt{-4}$.	15. $2\sqrt{-8}$.

187. Graphical representation of a complex number. The complex number $2+3\sqrt{-1}$ consists of a real part, 2, and the imaginary part, $3\sqrt{-1}$. To represent such a number graphically we measure, as in the figure on page 383, 2 units along OX from O to 2, and then 3 units parallel to the

imaginary axis II' upward from 2. This gives the point A, which is the graphical representation of the complex number, $2 + 3\,i$.



If the student pays proper attention to signs, he should see that the point B corresponds to the number $2 - 3\,i$; the point C to $-4 + 2\,i$; and the point D to $-5 - i$.

In general, if x is a complex number of the form $a + bi$, x is represented by a point a units from the imaginary axis and b units from the real axis, the positive and negative directions being those employed as in previous graphical work.

EXERCISES

Represent graphically the following complex numbers:

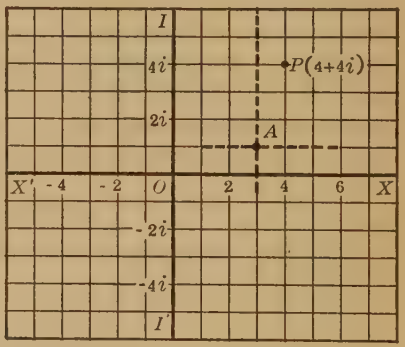
1. $2 + 3\,i$.
2. $3 - 2\,i$.
3. $-4 + i$.
4. $-3 - 2\,i$.
5. $5 - i$.
6. $2 - 3\,i$.
7. $-1 + 4\,i$.
8. $-2 - 3\,i$.
9. $5 + 3\sqrt{-1}$.
10. $-3 + \sqrt{-4}$.
11. $5 \pm \sqrt{-8}$.
12. $3 + \sqrt{3}\sqrt{-12}$.
13. $2 + \sqrt{-11}$.
14. $-2 \pm \sqrt{-\frac{1}{4}}$.
15. $\sqrt{-12} - 5$.

Graph separately each of the following factors, then graph their product:

16. $(-4 + \sqrt{-1})(2 + \sqrt{-4})$.
17. $(-1 - 3\,i)(1 + 5\,i)$.
18. $(-1 + \sqrt{-9})(2 - 2\sqrt{-4})$.

NOTE. The discovery of a graphical interpretation for the imaginary numbers did more than anything else to make them of significance to students of mathematics. Until this discovery imaginary numbers were tolerated because their appearance as roots of equations was a constant reminder of their existence. But such roots were usually regarded as meaningless, and the less one had to do with them the better. About 1800, a Norwegian by the name of Wessel and the Frenchman Argand gave practically the same graphical interpretation of imaginary numbers as that found in this text, but the work of these men was little noticed until Gauss adopted the method and, by his influence and ability, placed the imaginary number on a firm basis.

188. Graphical representation of the sum of two complex numbers. Since complex numbers are subject to all the laws of rational numbers, the principles involved in their graphical representation do not differ materially from those involved in the graphical representation of real numbers. To add graphically $(3+i)$ and $(1+3i)$, first locate the point A, $(3+i)$. Then add $(1+3i)$ to the value represented by the point A, $(3+i)$, by going 1 unit to the right of A and then 3 units upward, thus locating the point P, which represents graphically the sum of the complex numbers $(3+i)$ and $(1+3i)$, or $(4+4i)$.

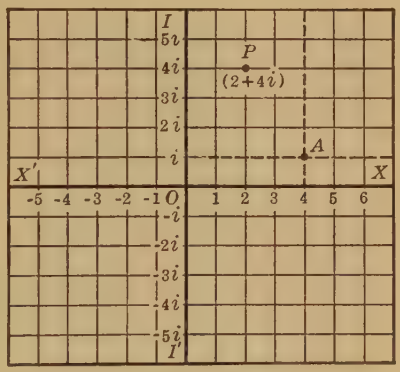


EXERCISES

Represent graphically:

1. $(1+2i)+(4+i)$.
2. $(2+i)+(-4+2i)$.
3. $(-2+i)+(-1-4i)$.
4. $(3-i)-3i$.
5. $(2+i)+(-3+i)$.
6. $(3+3i)+(-2+2i)$.

189. Graphical representation of the difference of two complex numbers. The method of representing the difference of two complex numbers graphically is very similar to the method of representing the sum of two complex numbers. For example, to subtract $(2-3i)$ from $(4+i)$ is equivalent to adding $(-2+3i)$ to $(4+i)$ and may be performed as follows. First locate the point A, $(4+i)$. To add $(-2+3i)$ to the value represented by point A, $(4+i)$, proceed as in addition, going 2 units to the left of A and then 3 units upward, thus locating the point P, which represents graphically the difference of $(2-3i)$ and $(4+i)$, or $(2+4i)$.



EXERCISES

Represent graphically:

1. $(-1+3i)-(3+2i)$.
2. $(4+i)-(1+3i)$.
3. $(3-2i)-(5-i)$.
4. $(2+3i)-(1+4i)$.
5. $5-(-1-6i)$.
6. $-6-(-1-3i)$.

EXERCISES

Solve the following equations, and represent their roots graphically:

1. $x^2-2x+5=0$.
2. $x^2+9=0$.
3. $x^3+1=0$.
4. $2x^2-14x+37=0$.
5. $2x^2+6x+5=0$.
6. $x^2+x\sqrt{-64}-16=0$.

7. $x^4 + 12x^2 + 11 = 0$.

8. $x^2 + 7x + 18.5 = 0$.

9. $4x^2 + 9 = 0$.

10. $x^2 + 6ix - 9 = 0$.

11. $x^4 + 1 = 0$.

12. $x^2 - 4x + 20 = 0$.

NOTE. The graphical interpretation of the definition of equality of complex numbers is that equal complex numbers are represented by the same point in the plane.

The student should not conclude from the preceding exercises that the imaginary axis has merely replaced the y-axis of our previous graphical work, for such is not the case. We have simply used our device of rectangular axes to represent complex numbers, in place of real numbers. The fact that we have previously made use of the same means to graph a function should not disturb us any more than it would to play baseball on a football field. The fact that both take place on the same field does not affect the game being played.

For some purposes it is convenient to relate the xy-plane, in which real number relations are represented, and the complex plane more closely to each other. In fact, we may proceed as if we had merely added a third axis, the axis of imaginary numbers, to the two axes of real numbers which we had before. This axis may be regarded as passing through the origin at right angles to both the x-axis and the y-axis. By means of it we may locate such a point as $y = 4$, $x = 3 + 3\sqrt{-1}$, or any point, in fact, in which one coördinate is real and the other is imaginary or complex. Then we may go on and construct the imaginary branches of many of those curves, the real graphs of which the student has already drawn.

NOTE ON THE USE OF COMPLEX NUMBERS. We have seen in section 186 that multiplication by an imaginary number can be considered as producing a rotation of 90° about the origin as a center. It was stated that a rotation through any other angle could be produced by using a properly chosen complex factor. We are now able to give an illustration of the case discussed in section 186.

If we multiply a real number by a complex number of the form $a + ai$, we will have caused the real number to be moved, counterclockwise, through 45°. Other complex factors of the more general form $a \pm bi$ will produce rotation in either direction through any angle. The student, after a little study, should now be able to determine the values of a and b in $a \pm bi$ for any angle of rotation.

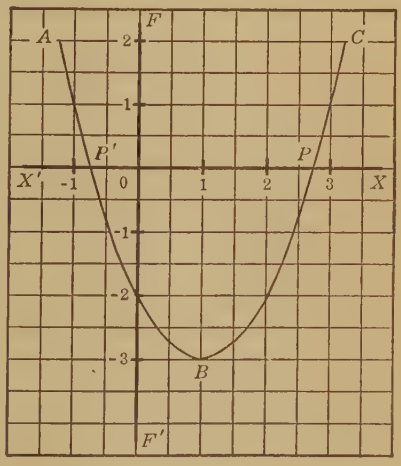
190. Greater accuracy in graphical solutions. In the graphical solutions of equations (Chapter XII) we have been able to determine the roots by a rough approximation. We can, however, approximate the values of the roots more closely by drawing to a larger scale that portion of the graph where the curve crosses the x-axis. This is shown in the graphical solution of the quadratic equation

$$x^2 - 2x - 2 = 0.$$

For $x^2 - 2x - 2 = 0$

If $x =$	-1	$-\frac{1}{2}$	0	$+1$	$+2$	$+3$
then $f(x) = x^2 - 2x - 2 =$	$+1$	$-\frac{3}{4}$	-2	-3	-2	$+1$

Locating the points corresponding to the pairs of numbers in the table, we obtain the curve ABC, as shown in the accompanying graph.

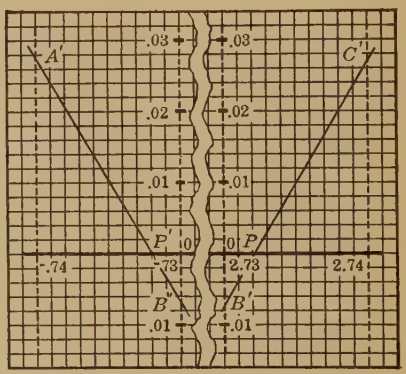


From the adjoining figure we see that the curve crosses the x-axis at two points, P and P'. There are, then, two real values of x which will make the function equal zero. These roots are approximately, from the graph, $+2.7$ and $-.7$.

When $x = 2.7$, the function, $x^2 - 2x - 2$, equals $-.11$. This result, being negative, we see from the graph that the root must be larger than 2.7. Hence, we substitute $x = 2.71$, and the function becomes

$-.0759$, which is still negative and indicates a still larger root. Substituting $x = 2.73$, the function equals $-.0071$. If $x = 2.74$, the function this time equals $+.0276$, or is positive. Thus we see that the curve crosses the x-axis, giving a value of x somewhere between 2.73 and 2.74.

We can find this value with greater accuracy by making the side of each small square, in the diagram given below, equal to .001 and locate the points B', (2.73, $-.0071$) and C', (2.74, $+.0276$). Since very small portions of the curve are almost straight lines, we join the two points by a straight line $B'C'$ which intersects the x-axis at a point which we estimate to be $x = +2.732$.

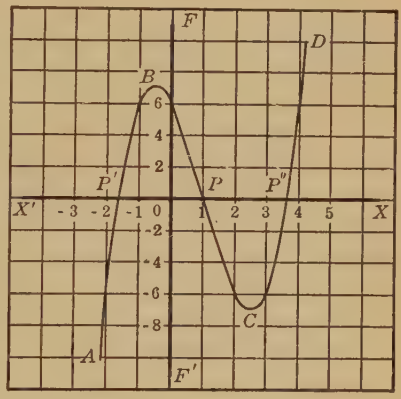


In a similar way the second root can be approximated from the left-hand side of the diagram, and it is found that $x = -.732$. These values can easily be verified by substitution into the equation and also by comparison with the roots, $x = 1 \pm \sqrt{3}$, obtained from the algebraic solution.

As previously stated, the graphic solution of cubic and higher equations is, whenever the method of factoring fails, the only method open to the student at this time. The graphical solution of the equation, $x^3 - 3x^2 - 4x + 6 = 0$, will be given in the following:

If $x =$	-2	-1	0	1	2	3	4	$-\frac{1}{2}$	$+2\frac{1}{2}$
then $f(x) = x^3 - 3x^2 - 4x + 6 =$	-6	$+6$	$+6$	0	-6	-6	$+6$	$+7\frac{1}{8}$	$-7\frac{1}{8}$

Plotting the points corresponding to the pairs of numbers in the table, we get the curve of the adjoining figure. Since the curve crosses the x-axis in three points, P', P, and P'', the equation has three real roots. From the graph we see that one root, represented by point P, is exact, $x = 1$. The other two roots, represented by P' and P'', are inexact and therefore have to be approximated by the method used in approximating the roots of a quadratic from its graph.

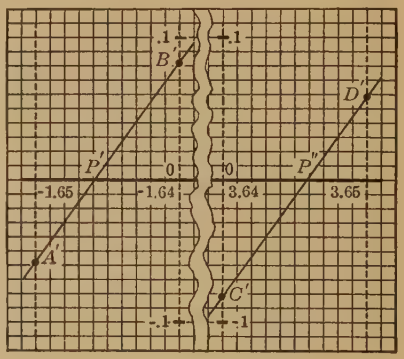


From the figure we find that the value of x, OP', is approximately -1.6. Computing the values of the function near $x = -1.6$, we have the following table:

If $x =$	-1.6	-1.64	-1.65
then $f(x) = x^3 - 3x^2 - 4x + 6 =$	$+.624$	$+.0805$	$-.05963$

From this table we see that when $x = -1.6$, the function $x^3 - 3x^2 - 4x + 6$ equals $+.624$. This result being positive, we see from the graph that the root must have a greater absolute value than -1.6. Hence we substitute $x = -1.64$ and the function becomes $+.0805$, which is still positive and indicates a root of still greater absolute value. When $x = -1.65$, the function becomes $-.059625$, and is negative. Thus, the curve crosses the x-axis and gives a value of x somewhere between -1.64 and -1.65.

We can find this value with greater accuracy by drawing the curve, near P', to a larger scale, as shown in the left-hand side of the adjacent diagram, locating the points A', $(-1.64, +.0805)$ and B', $(-1.65, -.059625)$. Since very small portions of the curve are almost straight lines, we will make no appreciable error by assuming that they are absolutely straight lines for the very short distances under consideration here. We may therefore join the two points A' and B' by a straight line which intersects the x-axis at the point P', from which we estimate x to be -1.646.



Again from the figure we find that another value of x, which makes the function zero, is $+3.6$ approximately. Computing values of the function near $x = +3.6$ we have the following table:

If $x =$	$+3.6$	$+3.64$	$+3.65$
then $f(x) = x^3 - 3x^2 - 4x + 6 =$	$-.624$	$-.0803$	$+.05963$

The graphical representation of the points in the above table, between which the curve crosses the x-axis, appears at the right in the above diagram. From this graph we are able to estimate that the x value of the intersection is $+3.646$, giving 3.646 as a root of the equation. Therefore the three roots of $x^3 - 3x^2 - 4x + 6 = 0$ are -1.646, $+1$, and $+3.646$.

EXERCISES

Graph the following equations and estimate the real roots correct to the nearest thousandth:

1. $2x^2 - x - 7 = 0$.
2. $3x^2 - 2x - 2 = 0$.
3. $x^3 - 2x^2 - 3x + 6 = 0$.
4. $x^3 + 2x^2 - 2x - 4 = 0$.
5. $x^3 - 2x^2 - 4x + 5 = 0$.
6. $x^4 - 5x^3 + 10x^2 + 10x - 24 = 0$.

191. Maxima and minima. In many practical problems it is important to determine the value of the variable for which the function takes on its greatest (or least) possible value for the range of values under consideration. Such problems can often be solved graphically with sufficient exactness for most purposes. The method of solution can be made clear by reference to the graph on page 181. There $f(x) = x^3 - 7x + 2$. Suppose we wish to know the value of x which gives $x^3 - 7x + 2$ the greatest possible value between -4 and -1. Near C on the curve appears a high point which is a turning point. Here x equals about -1.5, and $f(x)$ approximately 10.0.

This value of x affords a greater value of the function than any other between -4 and -1. In fact this is the greatest value that the function takes on for any value except those at the extreme right of the curve. In practical problems some feature of the actual situation indicates the range of values within which the greatest or least value is desired.

CHAPTER XXVII

SUPPLEMENTARY EXERCISES

MULTIPLICATION

Multiply:

1. $3x^2 - 8x - 2$ by $3x^2 + 8x + 2$.
2. $a^3 - 3a^2 + 3a - 2$ by $3a^4 - a^2 + 1$.
3. $2a + b - 3c$ by $3a + 5b - 2c$.
4. $x^4 + 2x^3y + 2x^2y^2 + 4xy^3 + 4y^4$ by $x^4 - 2x^3y + 2x^2y^2 - 4xy^3 + 4y^4$.
5. $x^{2a} + 3x^a - 5$ by $2x^{2a} + x^a - 4$.
6. $e^{2x} + 2 + e^{-2x}$ by $e^{2x} - 2 + e^{-2x}$.
7. $4x^{3a} - 5x^{2a} + 6$ by $3x^{2a} - 4x^a - 10$.
8. $x^{2a} - x^a - 5 + x^{-a}$ by $2x^a - 6 + 5x^{-a}$.
9. $x^{\frac{2}{3}} + y^{\frac{2}{3}}$ by $x^{\frac{4}{3}} - x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$.
10. $x^2 + 2x^{-2} + 3x^{-6}$ by $x^4 - 2 + x^{-4}$.
11. $3a^{\frac{1}{2}} - 5 + 2a^{-\frac{1}{2}}$ by $3a^{\frac{1}{2}} + 2a^{-\frac{1}{2}} + 5$.
12. $3x^a - 7x^{-\frac{a}{2}} + 5x^{-2a}$ by $3x^a + 7x^{-\frac{a}{2}} + 5x^{-2a}$.
13. $x^{\frac{1}{2}} + x^{\frac{1}{4}}a^{\frac{1}{4}} + a^{\frac{1}{2}}$ by $x^{\frac{1}{2}} - x^{\frac{1}{4}}a^{\frac{1}{4}} + a^{\frac{1}{2}}$.
14. $x + 2y + 5$ by $x^2 - 5x - 2xy + 25 - 10y + 4y^2$.
15. $x^2 - xy - xz + y^2 - yz + z^2$ by $x + y + z$.
16. $x^3 + x^2y + xy^2 + y^3$ by $x - y$.
17. $a^3 + a^2b + ab^2 + b^3$ by $a^3 - a^2b + ab^2 - b^3$.
18. $m^4 - m^3n + m^2n^2 - mn^3 + n^4$ by $m + n$.

DIVISION

Divide:

1. $2x^4 - 13x^3 + 23x^2 - x - 60$ by $x^2 - 3x - 5$.

2. $a^4 - 7a^2x^2 + 81x^4$ by $a^2 + 9x^2 - 5ax$.

3. $x^4 + 12ax - 36a^2 - x^2$ by $x - 6a - x^2$.

4. $19m + 2m^5 - 6m^3 - 7m^2 - 3m^4 + 20$ by $3m + 4 - 2m^2$.

5. $x^8 - 4x^4y^4 + 16y^8$ by $x^4 + 4y^4 + 2x^2y^2 + 4xy^3 + 2x^3y$.

6. $2 + 3a^5 - 5a - a^4 + 9a^2 - 8a^3$ by $3a^3 + 5a^2 - a + 2$.

7. $c^2 + 2cd + d^2 - c - d + \frac{1}{4}$ by $2c + 2d - 1$.

8. $64 + a^{4x+8}$ by $a^{2x+4} - 4a^{x+2} + 8$.

9. $e^{3x} + 54e^{2x} + 81e^{-4x} - 12e^{5x} - 108e^{-x}$ by $e^{4x} + 9e^{-2x} - 6e^{x}$.

10. $a^4 + 225a^{-4} - 39$ by $3 + 15a^{-2} - a^2$.

11. $9x^2 - 6 + 4x^{-4} + 12x^{-1} - 4x^{-3} + x^{-2}$ by $3x + 2x^{-2} - x^{-1}$.

12. $4 + 9x^3 + x^{-1} - 12x^2 - 2x$ by $3x - x^{-1} - 2$.

13. $a^3 + 8m^3 + 27 - 18am$ by $a + 2m + 3$.

14. $a^8 - 4a^4c^4 + 16c^8$ by $a^4 - 2a^3c + 4c^4 - 4ac^3 + 2a^2c^2$.

15. $8x^{-n} - 3x^{-3n} + 5x^{3n} - 8x^n$ by $5x^n - 3x^{-n}$.

16. $1 - \sqrt{n} + 2n^2 - \dfrac{2}{n^{-1}}$ by $1 - n^{\frac{1}{2}}$.

17. $m^{\frac{2}{3}} - m^{-\frac{1}{3}} - 1 + m$ by $m^{-\frac{1}{3}} + 1 + m^{\frac{1}{3}}$.

18. $2m^{\frac{2}{3}}n^{-1} + 32m^{-2}n^3 - 2m^{-\frac{2}{3}}n$ by $-6m^{-\frac{2}{3}}n + 2 + 8m^{-\frac{4}{3}}n^2$.

19. $\left(\dfrac{a^{6x} - 1}{a^{3x}} - \dfrac{3a^{2x} - 3}{a^x}\right)$ by $a^x - a^{-x}$.

20. $\left(\dfrac{a^{10x} + 54a^{2x} + 81a^{-6x}}{a^{2x}} - \dfrac{12a^{9x} + 108a^x}{a^{5x}}\right)$ by $a^{4x} + 9a^{-4x} - 6$.

SOLUTION OF EQUATIONS BY FACTORING

Solve (by factoring if possible) and check:

1. $x^2 + 7x = 8$.
2. $3x^2 + 2 = 5x$.
3. $x^3 - 5x^2 + 6x = 0$.
4. $x^3 - 16x = 0$.
5. $a^2x^2 + x^2 = a^2x + x$.
6. $a^2x + 2x^2 = ax^2 + 2a^2$.
7. $2m^2x + x^2 = 2mx^2 + m^2$.
8. $x^3 - 6x^2 + 11x - 6 = 0$.

HINT. Use the factor theorem.

9. $x^3 - 5x^2 + 8x - 4 = 0$.
10. $x^3 - 7x^2 + 16x = 12$.
11. $x^3 - 4x^2 + x + 6 = 0$.
12. $x^3 - 13x^2 + 52x = 60$.
13. $x^3 + 24 = x(x + 14)$.
14. $12 + x^3 = x(3x + 4)$.
15. $x^3 - x + 4 = 4x^2$.
16. $x^4 - 5x^2 + 4 = 0$.
17. $x^4 + 36 = 13x^2$.
18. $x^3 - 8 = 0$.
19. $x^3 + 27 = 0$.
20. $x^4 + 2x^2 + 1 = 0$.
21. $8x^3 = 1$.
22. $x^3 = 64$.
23. $x^4 = 16$.
24. $x^3 - (a + 2)x^2 + (2a + 1)x - a = 0$.
25. $x^3 - (m + 4)x^2 + (4m + 4)x - 4m = 0$.
26. $x^3 - mx^2 - 4x + 4m = 0$.
27. $x^3 - 2x^2 - m^2x - 2mx - x + 2(m + 1)^2 = 0$.
28. $x^3 - (2n + 2)x^2 + (2m - 2n)x + 4(n + 1)(n - m) = 0$.
29. $\dfrac{x^2}{a^2} = \dfrac{x + 3}{a + 3}$.
30. $\dfrac{x^2}{a^2} = \dfrac{2x - 3}{2a - 3}$.
31. $\dfrac{x^2}{x^2 - 2x + 4} = \dfrac{a^2}{a^2 - 2a + 4}$.
32. $x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0$.
33. $x^3 + ax^2 - bx^2 + cx^2 - bcx - abx + acx - abc = 0$.

FRACTIONS

(Addition and Subtraction)

Find the values of the following expressions:

1. $\dfrac{3\,m - n \pm (m + n)}{2}$.

2. $\dfrac{c - mk \pm (mk + c)}{2\,m}$.

3. $\dfrac{-2 \pm (4\,m + 2)}{2\,m^2 + 2\,m}$.

4. $\dfrac{a^2 \pm (a^2 + 8\,a)}{2\,a + 8}$.

5. $\dfrac{-h - 4 \pm (5\,h - 4)}{2\,h^2 - 4\,h}$.

6. $\dfrac{(2\,a)^2 \pm 4\,(a - 3)^2}{4\,a - 6}$.

7. $\dfrac{am^2 \pm (am^2 - 2\,cm)}{2\,am - 2\,c}$.

8. $\dfrac{-cm + c + 5 \pm (cm - c + 5)}{2\,c}$.

9. $\dfrac{n^2 + 3\,n + 2 \pm (n^2 - 3\,n - 2)}{2\,n}$.

10. $\dfrac{-4\,m - hm - 3\,h + 12 \pm (4\,m - hm - 3\,h - 12)}{2\,m^2 - 18}$.

11. $\dfrac{-4\,m \pm \sqrt{16\,m^2 + 84\,m^2}}{-14}$.

12. $\dfrac{7\,r \pm \sqrt{49\,r^2 - 48\,r^2}}{12\,ar^2}$.

13. $\dfrac{3\,m \pm \sqrt{9\,m^2 + 616\,m^2}}{44}$.

14. $\dfrac{-3 + a \pm \sqrt{9 - 6\,a + a^2 + 12\,a}}{am}$.

15. $\dfrac{-2 \pm \sqrt{4 + 16\,m^2 + 16\,m}}{2\,m^2 + 2\,m}$.

16. $\dfrac{a^2 \pm \sqrt{a^4 + 4\,a^2c^2 - 4\,a^3c}}{2\,ac - 2\,a^2}$.

17. $\dfrac{-a-4\pm\sqrt{a^2+8a+16+24a^2-48a}}{2a^2-4a}$.

18. $\dfrac{2a-3\pm\sqrt{4a^2-20a+25}}{2a^2-4a}$.

19. $\dfrac{3a^2\pm\sqrt{9a^4-4a^2m(3a-m)}}{6a-2m}$.

20. $\dfrac{am^2\pm\sqrt{a^2m^4-4m^2c(am-c)}}{2amn-2cn}$.

21. $\dfrac{6a\pm\sqrt{36a^2-24a^2}}{6}$.

22. $\dfrac{6a-2\pm\sqrt{3a^2-12a+12}}{1}$.

23. $\dfrac{6a-4\pm\sqrt{8m^2-48am+72a^2}}{4n}$.

24. $\dfrac{10a\pm2\sqrt{3a^2-12a+12}}{5a-2\sqrt{3}+a\sqrt{3}}$.

25. $\dfrac{na+2n\pm n\sqrt{a^3+3a^2-4}}{a+2-(a+2)\sqrt{a-1}}$.

26. $\dfrac{3m^2\pm\sqrt{9m^4-4am^2(3m-a)}}{6m-2a}$.

27. $\dfrac{3m^2-3\pm\sqrt{(3m^2+3)^2-4(3m)(3m)}}{6m}$.

28. $\dfrac{n^2+3n+2\pm\sqrt{n^4-6n^3+5n^2+12n+4}}{2n}$.

29. $\dfrac{a^2}{2a+4}\pm\sqrt{\left(\dfrac{a^2}{2a+4}\right)^2+\dfrac{2a^2}{a+2}}$.

30. $-\dfrac{m^2}{2m-6}\pm\sqrt{\dfrac{-3m^2}{m-3}+\left(\dfrac{m^2}{2m-6}\right)^2}$.

31. $-\dfrac{3n^2}{6n-14}\pm\sqrt{\dfrac{-7n^2}{3n-7}+\left(\dfrac{3n^2}{6n-14}\right)^2}$.

32. $\dfrac{ct}{c-at} \pm \sqrt{\left(\dfrac{at^2}{2\,at-2\,c}\right)^2 - \dfrac{ct^2}{at-c}}$.

33. $\dfrac{am^2}{2\,am+2\,c} \pm \sqrt{\dfrac{m^2c}{am+c} + \dfrac{1}{4}\left(\dfrac{am^2}{am+c}\right)^2}$.

34. $-\dfrac{5-3\,c}{4-2\,c} \pm \sqrt{\left(\dfrac{5-3\,c}{4-2\,c}\right)^2 - \dfrac{3}{c-2}}$.

COMPLEX FRACTIONS

(Multiplication and Division)

Simplify:

1. $\dfrac{\dfrac{y-x}{y^2}}{1+\left(\dfrac{x}{y}\right)^2}$.

2. $\dfrac{\dfrac{1}{a}}{1+\left(\dfrac{x-a}{a}\right)^2}$.

3. $\dfrac{-\,a(x+a)^{-2}}{1+\left(\dfrac{a}{x+a}\right)^2}$.

4. $\dfrac{\dfrac{cx(a)-(ax+b)c}{(cx)^2}}{1+\left(\dfrac{ax+b}{cx}\right)^2}$.

5. $\dfrac{\dfrac{x^n(2\,x)-x^2(nx^{n-1})}{(x^n)^2}}{1+\left(\dfrac{x^2}{x^n}\right)^2}$.

6. $\dfrac{\dfrac{a+m}{a}-\dfrac{a-m}{m}}{\dfrac{a-m}{a}+\dfrac{a+m}{m}}$.

7. $\dfrac{\dfrac{a}{m}+\dfrac{a-m}{a+m}}{\dfrac{a}{m}-\dfrac{a-m}{a+m}}$.

8. $\dfrac{\dfrac{a^2-5\,am-14\,m^2}{a^2+5\,am-24\,m^2}}{\dfrac{a^2-3\,am-28\,m^2}{a^2-8\,am+15\,m^2}}$.

9. $\dfrac{3-\dfrac{a-5\,m}{a-2\,m}}{5-\dfrac{a^2-19\,m^2}{a^2-4\,m^2}}$.

10. $\dfrac{\dfrac{(3\,m+2\,a)^2}{3\,m}-8\,a}{\left(\dfrac{4\,a+3\,m}{3\,m}\right)^2-\dfrac{6\,a}{m}}$.

11. $\dfrac{\dfrac{(ax-1)a-(ax+1)a}{(ax-1)^2}}{1+\left(\dfrac{ax+1}{ax-1}\right)^2}$.

12. $\dfrac{1+\dfrac{2}{m}+\dfrac{m}{2}}{m^2-\dfrac{8}{m}}$.

13. $\dfrac{\dfrac{a^2+c^2}{a}-2c}{1-\dfrac{c}{a}}$.

14. $\dfrac{\left(\dfrac{1}{n}-\dfrac{3}{m}\right)^2}{\dfrac{(m+3n)^2}{6mn}-2}$.

15. $\dfrac{\dfrac{(5a+4c)^2}{4a}-20c}{\left(\dfrac{5a+4c}{4a}\right)^2-\dfrac{5c}{a}}$.

16. $\dfrac{2+\dfrac{m}{n}}{\dfrac{(m-2n)^2}{4mn}+2}$.

17. $\dfrac{8m-\dfrac{(3m+2c)^2}{3c}}{\left(\dfrac{3m+4c}{3m}\right)^2-\dfrac{6c}{m}}$.

18. $\dfrac{\dfrac{(1-mx)-(x+m)(-m)}{(1-mx)^2}}{1+\left(\dfrac{x+m}{1-mx}\right)^2}$.

19. $\dfrac{\dfrac{x^{2(n+1)}x^n-x^{n+1}(2x)}{(x^2)^2}}{\dfrac{x^{n+1}}{x^2}}$.

20. $\dfrac{\dfrac{x^{2n}(-3x^{-4})-x^{-3}(2nx^{2n-1})}{(x^{2n})^2}}{\dfrac{x^{-3}}{x^{2n}}}$.

21. $\dfrac{\dfrac{x^n(ax^{a-1})-x^a(nx^{n-1})}{(x^n)^2}}{\dfrac{x^a}{x^n}}$.

22. $\dfrac{\dfrac{x^{3a}(2nx^{2n-1}) - x^{2n}(3ax^{3a-1})}{(x^{3a})^2}}{1+\left(\dfrac{x^{2n}}{x^{3a}}\right)^2}.$

23. $\dfrac{\dfrac{e^x - e^{-x}}{2}}{1+\left(\dfrac{e^x + e^{-x}}{2}\right)^2}.$

24. $\dfrac{\dfrac{e^{2x}(e^x - e^{-x}) - (e^x + e^{-x})2e^{2x}}{(e^{2x})^2}}{\dfrac{e^x + e^{-x}}{e^{2x}}}.$

25. $\dfrac{\dfrac{(bx+m)m - (mx+b)b}{(bx+m)^2}}{1+\left(\dfrac{mx+b}{bx+m}\right)^2}.$

26. $\dfrac{\dfrac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{(e^x + e^{-x})^2}}{\dfrac{e^x - e^{-x}}{e^x + e^{-x}}}.$

27. $\dfrac{\dfrac{(e^v - e^{-v})^2 - (e^v + e^{-v})^2}{(e^v - e^{-v})^2}}{1+\left(\dfrac{e^v - e^{-v}}{e^v + e^{-v}}\right)^2}.$

28. $\dfrac{\dfrac{(e^{ax} + e^{-ax})(ae^{ax} + ae^{-ax}) - (e^{ax} - e^{-ax})(ae^{ax} - ae^{-ax})}{(e^{ax} + e^{-ax})^2}}{\left(\dfrac{e^{ax} + e^{-ax}}{e^{ax} - e^{-ax}}\right)^2 + 1}.$

LINEAR EQUATIONS IN ONE UNKNOWN

(With Literal Coefficients)

Solve:

1. $mx + n^2 = nx + m^2$.

2. $(m^2 + x)(m^2 - x) = m^4 + 2mx - x^2$.

3. $(x - m)^2 + (n - c)^2 = (n + c)^2 + (m + x)^2$.

4. $(2x + c)(2a - 3x) = (2a - 3c - 3x)(2x - 2c)$.

5. $\frac{m}{n} + \frac{n}{m} = 2 - \frac{1}{mx} + \frac{1}{nx}$.

6. $\frac{4x}{n} - \frac{m}{2x} = \frac{4x^2 - m}{2nx} - \frac{m - 2x}{n}$.

7. $mn^2 + 2mx - \frac{5m^2 - 12nx}{2m} = \frac{4m + 2nx}{4} - 2nx - \frac{m^2 - 6nx}{m}$.

8. $\frac{a}{a - x} = \frac{c}{c - x}$.

9. $\frac{m}{n} = \frac{2x - n^2}{2x - m^2}$.

10. $\frac{c}{mx} + \frac{a}{nx} = m - n$.

11. $\frac{a + x}{ax} - \frac{1}{a} = \frac{a}{a + x}$.

12. $\frac{2 - ax}{ax - 3} + \frac{m^2 + n^2}{mn} = \frac{amx - n}{anx} + \frac{anx + m}{amx} + 2$.

13. $\frac{n}{4x^2 - m^2} = \frac{m}{2x - m} - \frac{n}{2x + m}$.

14. $\frac{2mx + n}{2x - 3} - \frac{2mx - n}{2x + 3} = \frac{5}{9 - 4x^2}$.

15. $\frac{mx - a - c}{nx - e - r} = \frac{mx - a - e}{nx - c - r}$.

16. $\frac{m}{x} + \frac{x}{m} - \frac{x(x + m)}{m(x - m)} = \frac{mx}{m^2 - x^2} - 2 - \frac{m(x - m)}{x(x + m)}$.

LINEAR EQUATIONS IN MORE THAN ONE UNKNOWN

Solve: (With Literal Coefficients)

1. $x + y = a,$
   $x - y = c.$

2. $ax + cy = n,$
   $x - y = m.$

3. $ax + by = m,$
   $bx + ay = n.$

4. $ax + by = 0,$
   $lx + my = n.$

5. $mx + ny = m^2 + n^2,$
   $x + y = m + n.$

6. $x + y = m,$
   $(n - c)x = (n + c)y.$

7. $nx + cx - ny - cy = 0,$
   $x + y = a.$

8. $\frac{2x}{a} + \frac{3y}{c} = 1,$
   $\frac{2x}{3a} + \frac{y}{2c} = \frac{2}{3}.$

9. $anx - rc = r(e - y),$
   $\frac{anx}{e} - r\left(1 + \frac{y}{c}\right) = -r.$

10. $mx + ny = a,$
    $cx + ey = r.$

11. $(b + c)y + anx = 1,$
    $ay = 1 - nx(b + c).$

12. $my(b - c) + ax = 1,$
    $amy = 1 - x(b - c).$

13. $\frac{x}{x - m} - \frac{y}{y - n} = 1,$
    $x + y = m + n.$

14. $\frac{m}{x} + \frac{n}{y} = 1,$
    $\frac{n}{x} - \frac{m}{y} = 1.$

15. $\frac{ax + cy}{m} = \frac{rx - ty}{n} = k.$

16. $\frac{x + a}{y - n} = \frac{m + n}{m + a},$
    $\frac{x + y}{x - y} = \frac{m + n}{-a}.$

17. $\frac{n}{m + x} + \frac{m}{n - y} = \frac{n}{m},$
    $\frac{m}{m + x} + \frac{n}{n - y} = \frac{m}{n}.$

18. $\frac{2x + 1}{ay + 1} = \frac{m + n + 1}{m - n + 1},$
    $2x - ay = 2n.$

19. $x + y = m,$
    $x + z = n,$
    $y + z = a.$

20. $mx + ny = 1,$
    $ny + cz = 1,$
    $cz + mx = 1.$

21. $xy = -18$,
$xz = 15$,
$yz = -30$.

22. $xyz = 600$,
$xyv = 240$,
$yzv = 360$,
$xzv = 900$.

23. $x + y = m$,
$x - z = n$,
$y - v = a$,
$v - z = c$.

24. $\frac{x}{a} = \frac{y}{m} = \frac{z}{n} = (x+y+z)^2$.

25. $\frac{ax+by}{l} = \frac{cx+dz}{m} = \frac{ey+fz}{n} = k$.

SQUARE ROOT

Extract the square root of:

1. $n^4 + 4n^3 - 2n^2 - 12n + 9$.
2. $m^6 + 4 + 20m - 10m^4 + 25m^2 - 4m^3$.
3. $4n^8 + 12n^4 - 7 - 24n^{-4} + 16n^{-8}$.
4. $x^{2a} + x^2 + x^{-2a} - 2 - 2x^{1-a} + 2x^{a+1}$.
5. $x^{2n} + 9x^4 + 4 - 6x^{n+2} + 4x^n - 12x^2$.
6. $x^{2n} + 2x^{n+a} - 2x^{n+1} + x^{2a} - 2x^{a+1} + x^2$.
7. $m^{2a-2} + m^{2a+2} + 2m^{2a+1} - m^{2a} - 2m^{2a-1}$.
8. $\frac{x^2}{9m^2} + \frac{16m^2}{x^2} + \frac{4x}{3m} + \frac{16m}{x} + \frac{20}{3}$.
9. $\frac{x^2}{4m^2} - \frac{3x}{m} + 9 + \frac{2}{5} - \frac{12m}{5x} + \frac{4m^2}{25x^2}$.
10. $n^4 + 9n^2 + 4 + 6n^3 + 4n^2 - 12n - 12n^3$.
11. $c^2m^2 + c^2 + 25 - 2c^2m - 10cm + 10c + 20cm - 20c$.
12. $16m^2 + h^2m^2 + 9h^2 + 144 - 8hm^2 - 96m + 6mh^2 + 72h$.
13. Simplify:

$$\frac{a - b - 4ab \pm \sqrt{(4ab - a + b)^2 - 4(2b)(2ab - 2a^2)}}{4abn - 4b^2n}$$

RADICAL EQUATIONS

Solve, check, and ~~reject extraneous roots~~:

1. $\sqrt{2x+6}+2=6.$
2. $\sqrt{2x+1}+\sqrt{x}=5.$
3. $\sqrt{5x+4}+\sqrt{x}=10.$
4. $\sqrt{4-7x}-\sqrt{3x}=2.$
5. $\sqrt{x+4}+\sqrt{2x-1}=6.$
6. $\sqrt{x+a^2}=a+\sqrt{x}.$
7. $x-c=\sqrt{3x^2-5cx+2c^2}.$
8. $\sqrt{x-2m}=\sqrt{x+2m}-\sqrt{2a}.$
9. $\sqrt{9x-m}+\sqrt{a}=3\sqrt{x}.$
10. $(\sqrt{x+7})(\sqrt{x-7})=x+1.$
11. $\dfrac{3m}{\sqrt{x+m}}-\sqrt{x}=\sqrt{x+m}.$
12. $\sqrt{ax+m}-\sqrt{ax-m}=\sqrt{4ax-2m}.$
13. $\sqrt{n}-\sqrt{m}+\sqrt{x-n}=\sqrt{x-m}.$
14. $\dfrac{\sqrt{x+m}-\sqrt{m-x}}{\sqrt{x+m}+\sqrt{m-x}}=a.$
15. $\dfrac{\sqrt{x}-\sqrt{m}}{6}=\dfrac{x-m}{\sqrt{x}-\sqrt{m}}-2\sqrt{m}.$
16. $\sqrt{2x+1}+\sqrt{x}=5.$
17. $\sqrt{5x+1}+\sqrt{x}=13.$
18. $\sqrt{46-7x}-\sqrt{3x}=2.$
19. $\sqrt{x+1}+\sqrt{2x-2}=4.$
20. $\sqrt{x+4}+\sqrt{2x+6}=7.$
21. $\sqrt{1+2x\sqrt{4x^2+12}}=1+2x.$
22. $\sqrt{7+3\sqrt{10x-16}}-4=0.$

23. $\sqrt{17+2\sqrt{3+x+\sqrt{x+7}}}-5=0.$

24. $\dfrac{\sqrt{x}+9}{\sqrt{x}-7}=\dfrac{\sqrt{x}+20}{\sqrt{x}-12}.$

25. $\dfrac{\sqrt{2\,x}+18}{\sqrt{2\,x}+2}=\dfrac{32}{\sqrt{2\,x}+6}+1.$

EXPONENTS AND RADICALS

Simplify:

1. $\dfrac{4^{-\frac{1}{2}}}{2^{-1}-8^{-1}}.$

2. $\dfrac{4^{-2}+2^{-2}}{4^{-2}-2^{-2}}.$

3. $\dfrac{4^{-3}+3^{-3}}{4^{-2}-3^{-2}}.$

4. $\dfrac{x^{-2}-m^{-2}}{x^{-1}+m^{-1}}.$

5. $\dfrac{x^{-2}-4\,a^{-2}}{x^{-1}+2\,a^{-1}}.$

6. $\dfrac{x^{-3}-y^{-3}}{x^{-1}-y^{-1}}.$

7. $\dfrac{8\,x^{-3}+a^{-3}}{2\,x^{-1}+a^{-1}}.$

8. $\dfrac{3\,m^{-1}-2\,n^{-1}}{27\,m^{-3}-8\,n^{-3}}.$

9. $\dfrac{2^m\cdot 2-2^{3+m}}{8\cdot 2^{2+m}}.$

10. $\dfrac{m^{-2}-m^{-1}+1}{m^{-4}+m^{-2}+1}.$

11. $3\cdot 7^0-1^4+(3\cdot 4)^0-2(\tfrac{1}{16})^{\frac{1}{2}}\div 8^{-\frac{1}{3}}+(\tfrac{4}{9})^{-\frac{1}{2}}.$

12. $\dfrac{4^{2n+1}}{2^{n+3}}.$

13. $\left(\dfrac{8^n}{12^{n+1}}\right)\left(\dfrac{6^{n+2}}{3^4}\right).$

14. $\dfrac{4^{a+1}}{2^a(4^{a-1})^a}\cdot\dfrac{(4^{a+1})^{a-1}}{8^{a+1}}+5.$

15. $\dfrac{(16)^n}{(6)^{2n}}\div\dfrac{(8)^{3n+1}}{(12)^{n-1}}.$

16. $\dfrac{\sqrt{1-x^2}-x(1-x^2)^{-\frac{1}{2}}(-2\,x)}{(\sqrt{1-x^2})^2}.$

17. $\dfrac{2\,v\sqrt{1-v^2}-v^2(1-v^2)^{-\frac{1}{2}}(-2\,v)}{(\sqrt{1-v^2})^2}.$

18. $\sqrt{ax}\left(\dfrac{a}{2}\right)(ax-1)^{-\frac{1}{2}}+\sqrt{ax-1}\left(\dfrac{a}{2}\right)(ax)^{-\frac{1}{2}}.$

19. $\sqrt{x^3+1}\cdot\frac{1}{2}(x^3-1)^{-\frac{1}{2}}(3x^2)+\sqrt{x^3-1}\cdot\frac{1}{2}(x^3+1)^{-\frac{1}{2}}(3x^2)$.

20. $\dfrac{2v\sqrt{v^2-a^2}-\dfrac{v^2}{2}(v^2-a^2)^{-\frac{1}{2}}(2v)}{(\sqrt{v^2-a^2})^2}$.

21. $\dfrac{\sqrt{a-mv^2}-\dfrac{v}{2}(a-mv^2)^{-\frac{1}{2}}(-2mv)}{(\sqrt{a-mv^2})^2}$.

22. $\sqrt{a+x}\cdot\frac{1}{2}(a-3x)^{-\frac{1}{2}}(-3)+\sqrt{a-3x}\cdot\frac{1}{2}(a+x)^{-\frac{1}{2}}$.

23. $\sqrt{x^3+1}\cdot\frac{1}{3}(x^2+1)^{-\frac{2}{3}}(2x)+\sqrt[3]{x^2+1}\cdot\frac{1}{2}(x^3+1)^{-\frac{1}{2}}(3x^2)$.

24. $\dfrac{v\sqrt{1+v^2}\cdot 4v-(2v^2)\left[\dfrac{v}{2}(1+v^2)^{-\frac{1}{2}}(2v)+\sqrt{1+v^2}\right]}{(v\sqrt{1+v^2})^2}$.

25. $\dfrac{\sqrt{1-3x}\cdot\frac{1}{2}(1+3x)^{-\frac{1}{2}}\cdot 3-\frac{1}{2}\sqrt{1+3x}(1-3x)^{-\frac{1}{2}}(-3)}{(\sqrt{1-3x})^2}$.

26. $\dfrac{\dfrac{1}{m}}{\sqrt{1-\dfrac{x^2}{m^2}}}$.

27. $\dfrac{\dfrac{x^2-(x+1)2x}{x^4}}{\dfrac{x+1}{x^2}\sqrt{\left(\dfrac{x+1}{x^2}\right)^2-1}}$.

28. $\dfrac{\dfrac{(v+1)2v-v^2}{(v+1)^2}}{\dfrac{v^2}{v+1}\sqrt{\left(\dfrac{v^2}{v+1}\right)^2-1}}$.

29. $\dfrac{\dfrac{(v^2+1)-v(2v)}{(v^2+1)^2}}{\dfrac{v}{v^2+1}\sqrt{\left(\dfrac{v}{v^2+1}\right)^2-1}}$.

30. $\dfrac{\dfrac{x^2nx^{n-1}-x^n(2x)}{(x^2)^2}}{\dfrac{x^n}{x^2}\sqrt{\left(\dfrac{x^n}{x^2}\right)^2-1}}$.

31. $\dfrac{\dfrac{x^2-(x+1)(2x)}{(x^2)^2}}{\sqrt{1-\left(\dfrac{x+1}{x^2}\right)^2}}$.

32. $\dfrac{\dfrac{(ax+1)a-ax(a)}{(ax+1)^2}}{\sqrt{1-\left(\dfrac{ax}{ax+1}\right)^2}}$.

33. $\dfrac{\dfrac{x^a\cdot 5x^4-x^5(ax^{a-1})}{(x^a)^2}}{1+\left(\dfrac{x^5}{x^a}\right)^2}$.

34. $\dfrac{\dfrac{x^4(3x^2)-(x^3+7)4x^3}{(x^4)^2}}{\dfrac{x^3+7}{x^4}}$.

35. $\dfrac{\dfrac{x[\frac{1}{2}(x+1)^{-\frac{1}{2}}]-\sqrt{x+1}}{x^2}}{1+\left(\dfrac{\sqrt{x+1}}{x}\right)^2}$.

36. $\dfrac{(x^2+x)3x^2-x^3(2x+1)}{(x^2+x)^2}\div\dfrac{x^3}{x^2+x}$.

37. $\dfrac{\dfrac{(1-x^2)(1+x^2)2x-(1+x^2)(1-x^2)(-2x)}{(1-x^2)^2}}{\dfrac{1+x^2}{1-x^2}}$.

38. $\dfrac{\dfrac{(x^3-1)nx^{n-1}-x^n(3x^2)}{(x^3-1)^2}}{\dfrac{x^n}{x^3-1}}$.

39. $\dfrac{\dfrac{(4+e^t)e^t-e^t\cdot e^t}{(4+e^t)^2}}{\dfrac{e^t}{4+e^t}}$.

40. $\dfrac{\dfrac{(e^t+e^{-t})^2-(e^t-e^{-t})^2}{(e^t+e^{-t})^2}}{\dfrac{e^t-e^{-t}}{e^t+e^{-t}}}$.

41. $\dfrac{(e^{ax}-e^{-ax})ae^{ax}-e^{ax}(ae^{ax}+ae^{-ax})}{(e^{ax}-e^{-ax})^2}\div\dfrac{e^{ax}}{e^{ax}-e^{-ax}}$.

42. $\dfrac{(x+1)\dfrac{a}{2}(ax)^{-\frac{1}{2}}-\sqrt{ax}}{(x+1)^2}\div\dfrac{\sqrt{ax}}{x+1}$.

43. $\dfrac{(x^2+a^2)^{\frac{1}{2}}nx^{n-1}-\dfrac{x^n}{2}(x^2+a^2)^{-\frac{1}{2}}(2x)}{(\sqrt{x^2+a^2})^2}\div\dfrac{x^n}{\sqrt{x^2+a^2}}$.

44. $\dfrac{x^n n(ax+b)^{n-1}a-(ax+b)^n nx^{n-1}}{x^{2n}}\div\dfrac{(ax+b)^n}{x^n}$.

45. $\dfrac{\dfrac{x^3(nx^{n-1}) - (x^n + 2)(3\,x^2)}{(x^3)^2}}{\dfrac{x^n + 2}{x^3}}$.

46. $\dfrac{\dfrac{v^{3a}(2\,nv^{2n-1}) - v^{2n}(3\,av^{3a-1})}{(v^{3a})^2}}{\dfrac{v^{2n}}{v^{3a}}}$.

47. $\dfrac{\dfrac{x^{-2}(-3\,x^{-4}) - (x^{-3} + 4)(-2\,x^{-3})}{(x^{-2})^2}}{\dfrac{x^{-3} + 4}{x^{-2}}}$.

48. $\dfrac{e^{nx}(-3\,ne^{-nx}) - (3\,e^{-nx} + 1)ne^{nx}}{(e^{nx})^2} \div \dfrac{3\,e^{-nx} + 1}{e^{nx}}$.

49. $\dfrac{\dfrac{\sqrt[3]{x} \cdot \frac{1}{3}\,x^{-\frac{2}{3}} - (\sqrt[3]{x} + 1)\frac{1}{3}\,x^{-\frac{2}{3}}}{(\sqrt[3]{x})^2}}{\dfrac{\sqrt[3]{x} + 1}{\sqrt[3]{x}}}$.

50. $\dfrac{(\sqrt[4]{x} + 1)\frac{1}{4}x^{-\frac{3}{4}} - \sqrt[4]{x} \cdot \frac{1}{4}\,x^{-\frac{3}{4}}}{(\sqrt[4]{x} + 1)^2} \div \dfrac{\sqrt[4]{x}}{\sqrt[4]{x} + 1}$.

QUADRATIC EQUATIONS

(With Literal Coefficients)

Solve for x:

1. $x^2 + 12\,n^2 = 7\,nx$.
2. $2\,a^2x^2 + 3 = 5\,ax$.
3. $\dfrac{x}{m} + \dfrac{m}{x} = \dfrac{m}{n} + \dfrac{n}{m}$.
4. $\dfrac{x}{m - x} - \dfrac{m - x}{x} = \dfrac{m}{x}$.
5. $\dfrac{2\,x + 1}{\sqrt{2\,x}} = \dfrac{a + 1}{\sqrt{a}}$.
6. $\dfrac{ax^2 - bx + c}{mx^2 - nx + r} = \dfrac{c}{r}$.
7. $\left(\dfrac{x + m}{x - m}\right)^2 + 10 = 7\left(\dfrac{x + m}{x - m}\right)$.

8. $\sqrt{4x} = \sqrt{m+2x} - \sqrt{m-2x}$.

9. $\sqrt{x-a} + \sqrt{m-x} = \sqrt{m-a}$.

10. $\sqrt{mx+n^2} - \sqrt{nx+m^2} = n-m$.

11. $\dfrac{m+x}{m-x} - \dfrac{m-x}{m+x} = \dfrac{35}{6}$.

12. $\dfrac{a}{x^2+x} + \dfrac{x^2+x}{a} = a$.

13. $\dfrac{x^2}{a^2} = \dfrac{x-10}{a-10}$.

14. $\dfrac{x^2}{2x-m} = \dfrac{a^2}{2a-m}$.

15. $\dfrac{x^2}{ax-b} = \dfrac{m^2}{am-b}$.

16. $\dfrac{x^2}{ax^2+bx+c} = \dfrac{m^2}{am^2+bm+c}$.

17. $\sqrt{2x-3a^2} - \sqrt{2x-2a^2} = \sqrt{4x-5a^2}$.

18. $\sqrt{x+m} - \sqrt{x-m} = \sqrt{2x}$.

19. $\sqrt{m-2x} + \sqrt{n-2x} = \sqrt{m+n-4x}$.

20. $\dfrac{m-\sqrt{2mx-x^2}}{m+\sqrt{2mx-x^2}} = \dfrac{x}{m-x}$.

THEORY OF QUADRATIC EQUATIONS

Form equations whose roots are the following:

1. $7, -16$.

2. $3, 2$.

3. $-5, -3$.

4. $\frac{2}{3}, \frac{1}{6}, \frac{1}{2}$.

5. $2\sqrt{-1}, -2\sqrt{-1}$.

6. $1-\sqrt{-1}, 1+\sqrt{-1}$.

7. $-4+\sqrt{2}, -4-\sqrt{2}, 4+\sqrt{2}, 4-\sqrt{2}$.

8. $-3-\sqrt{2}, -3+\sqrt{2}$.

9. $\frac{2}{3}+\sqrt{-6}, \frac{2}{3}-\sqrt{-6}$.

10. $\sqrt{b}, -\sqrt{b}$.

11. $-m+\sqrt{-p}, -m-\sqrt{-p}$.

12. $\frac{2}{3}, \frac{3}{2}$.

13. $\frac{1}{5}, \frac{7}{10}, \frac{2}{5}$.

14. $-7 \pm 2\sqrt{-3}$.

Find the value of the literal coefficient in the following:

15. $x^2+3ax-4=0$; if one root is 1.

16. $x^2-17x-b=0$; if one root is 5.

17. $x^2+2x+a=0$; if one root is 2 more than the other root.

SYSTEMS SOLVABLE BY QUADRATICS

Solve and check, as directed by the teacher:

1. $3x^2+2xy=65$,
   $2x+3y=7$.

2. $x^2-3y^2=-23$,
   $4x=9y+35$.

3. $7x^2-20y=23$,
   $2x=3y$.

4. $2y^2+13xy=-280$,
   $2x-y=16$.

5. $x^2+3xy+y^2=5$,
   $2x+y=3$.

6. $2x+3xy=-35$,
   $2x+3y=1$.

7. $x^2-y^2=3$,
   $3x^2+y^2=13$.

8. $2m^2+3n^2=27$,
   $5m^2-6n^2=-54$.

9. $x^2+xy=3a^2$,
   $xy+y^2=13a^2$.

10. $s^2+t^2=13$,
    $st=6$.

11. $\frac{1}{x^2}+\frac{1}{y^2}=\frac{10}{9}$,
    $\frac{1}{xy}=-\frac{1}{3}$.

12. $\frac{1}{x}-\frac{1}{y}=-\frac{5}{14}$,
    $\frac{1}{xy}=\frac{1}{14}$.

13. $x^2-y^2=-1$,
    $x+y=1$.

14. $16x^2-9y^2=0$,
    $4x-3y=0$.

15. $8x^3 - y^3 = 7,$
$4x^2 + 2xy + y^2 = 7.$

16. $4x^2 + y^2 = 20,$
$xy = -4.$

17. $x^2 - 9y^2 = -80,$
$x + 3y = -8.$

18. $2x + 3y = 7,$
$4x^2 - 9y^2 = -77.$

19. $-3xy = -18,$
$x^2 + y^2 = 13.$

20. $2x + y = 7,$
$xy = -15.$

21. $m + mn = 2,$
$n - mn = 0.$

22. $\dfrac{x+y}{2x-4} = 1,$
$\dfrac{x^2 - y^2}{2x^2 - 2x - 4} = \dfrac{2}{3}.$

23. $8x^3 + 125y^3 = 91,$
$4x^2 - 10xy + 25y^2 = 91.$

24. $x^2 + y^2 = 90,$
$5xy = 135.$

25. $\dfrac{1}{x^3} + \dfrac{1}{y^3} = \dfrac{19}{216},$
$\dfrac{1}{x} + \dfrac{1}{y} = \dfrac{1}{6}.$

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	52	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	57	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	58	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	61	3,721	226,981	7.810	3.937
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	68	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	69	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	71	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.885	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.963	76	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	78	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	81	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	84	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	86	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	92	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	95	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	97	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642

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$(a \pm b)^2 = a^2 \pm 2ab + b^2$

$(a+b)(a-b) = a^2 - b^2$

$(a \pm b)^3 = (a^3 \pm 3a^2b + 3ab^2 \pm b^3)$

$(a+b-c+d-e)^2 = a^2 + b^2 + c^2 + d^2 + e^2 + 2ab - 2ac$
$+ 2ad - 2ae - 2bc + 2bd - 2be$
$- 2cd + 2ce - 2de$

$(a+b-c)(a+b+c) = [(a+b)^2 - c^2] = a^2 + 2ab + b^2 - c$

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